

Homework 4 for Math 1540

Due day: Tuesday March 17, 2020 recitations

Problem 50. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be convex. Prove that if partial derivatives

$$\frac{\partial f}{\partial x_i}(x_0), \quad i = 1, 2, \dots, n,$$

exist, then f is differentiable at x_0 .

Proof. WRITE YOUR SOLUTION HERE. □

Problem 51. Let $Q : C^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$ be a linear mapping such that $Qf \geq 0$ whenever $f \in C^\infty(\mathbb{R}^n)$ satisfies $f(0) = 0$ and $f(x) \geq 0$ in a neighborhood of 0. Prove that there are real numbers a_{ij} , b_i and c such that

$$Qf = \sum_{i,j=1}^n a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(0) + \sum_{i=1}^n b_i \frac{\partial f}{\partial x_i}(0) + cf(0) \quad \text{for all } f \in C^\infty(\mathbb{R}^n).$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 52. Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{x^2(y^4+2x)}{x^2+y^4} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Prove that f is differentiable at $(0, 0)$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 53. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} xy \frac{x^2-y^2}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Prove that the mixed partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ exist everywhere in $\mathbb{R}^2$, but

$$\frac{\partial^2 f}{\partial x \partial y}(0, 0) \neq \frac{\partial^2 f}{\partial y \partial x}(0, 0).$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 54. Prove that the function

$$f(x) = \begin{cases} \frac{x|y|}{\sqrt{x^2+y^2}} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

has all directional derivatives $D_v f(0, 0)$ at the origin, but f is not differentiable at $(0, 0)$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 55. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a mapping of class C^1 . Prove that if $\text{rank } Df(x_0) = n$, then f is injective in a neighborhood of x_0 . Prove it directly, without using the inverse function theorem.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 56. Assume that $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable and $\text{rank } Df(x) = n$ for all $x \in \mathbb{R}^n$. Prove that if $S \subset \mathbb{R}^n$ is bounded, then for every $y \in \mathbb{R}^m$, the set

$$S \cap f^{-1}(y) = \{x \in S : f(x) = y\}$$

is finite.

Remark. Since we do not assume continuity of Df , we cannot use the inverse function theorem.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 57. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x > 0, 0 < y < 2\pi\}$. Prove that the mapping $f : \Omega \rightarrow \mathbb{R}^2$, $f(x, y) = (x \cos y, x \sin y)$ is a diffeomorphism of Ω onto an open subset of $\mathbb{R}^2$. Find $f(\Omega)$. *Hint:* A picture will help. Have you seen a similar mapping in Calculus 3?

Proof. WRITE YOUR SOLUTION HERE. □

Problem 58. Find a diffeomorphism of $\mathbb{R}^2$ onto the open unit disc $x^2 + y^2 < 1$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 59. Find a diffeomorphism of the upper half plane $y > 0$ onto the first quadrant $x > 0, y > 0$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 60. Suppose that $f \in C^1(\mathbb{R})$ is such that $|f'(x)| < 1$ for all $x \in \mathbb{R}$. Prove that the mapping $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $F(x, y) = (x + f(y), y - f(x))$ is a diffeomorphism in a neighborhood of any point $(x, y) \in \mathbb{R}^2$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 61. Prove that a complex polynomial $P(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_n$ regarded as a function $P : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a diffeomorphism in a neighborhood of $z_0 \in \mathbb{C}$ if and only if $P'(z_0) \neq 0$, where $P'(z) = na_0 z^{n-1} + (n-1)a_1 z^{n-2} + \dots + a_1$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 62. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be C^1 and let

$$\begin{aligned} u &= f(x) \\ v &= -y + xf(x). \end{aligned}$$

If $f'(x_0) \neq 0$, show that this transformation is locally invertible near (x_0, y_0) and the inverse has the form

$$\begin{aligned} x &= g(u) \\ y &= -v + ug(u). \end{aligned}$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 63. Prove that the system of equations

$$\begin{cases} xyz + x^2 + y = 0 \\ z + x^2 y^2 z^2 = 0 \end{cases}$$

has a solution of the form $y = y(x), z = z(x)$ in a neighborhood of $(0, 0, 0)$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 64. Let $F(x, y) = x^3 y^2 + 3x^2 y^3 - xy + 2x - y^2 + 1$, $(x, y) \in \mathbb{R}^2$. Prove that there exist functions $g, h \in C^\infty$ defined on an open neighborhood $U \subset \mathbb{R}$ of 0, such that $F(x, g(x)) = 0 = F(x, h(x))$ and $g(x) < h(x)$ for every $x \in U$. Find $g'(0)$, $h'(0)$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 65. Let F be as in Problem 64. Prove that there is a function $g \in C^\infty$ defined on an open neighborhood U of 0 such that $F(g(y), y) = 0$ for every $y \in U$. Find $g'(0)$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 66. Suppose that $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ is of class C^1 . $F(0, 0, 0) = 0$, $F_x(0, 0, 0) \neq 0$, $F_y(0, 0, 0) \neq 0$, $F_z(0, 0, 0) \neq 0$. The implicit function theorem implies that the equation $F(x, y, z) = 0$ can uniquely be solved in a neighborhood of the point $(0, 0, 0)$ as $x = x(y, z)$ or $y = y(x, z)$ or $z = z(x, y)$. Prove that at every point in some neighborhood of $(0, 0, 0)$ we have

$$\frac{\partial z}{\partial x} \frac{\partial x}{\partial y} \frac{\partial y}{\partial z} = -1.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 67. Let $F = F(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R}$ be of class C^1 such that $\partial F / \partial y \neq 0$ on $\mathbb{R}^2$. Prove that if the set $S = \{(x, y) \mid F(x, y) = 0\}$ is nonempty, then it is of the form $S = \{(x, g(x)) \mid x \in U\}$, where $g : U \rightarrow \mathbb{R}$ is a C^1 function defined on an open set $U \subset \mathbb{R}$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 68. Prove that the equation $xe^z = y(z + x)$ defines z as a function of (x, y) in a neighborhood of the point $(x_0, y_0, z_0) = (2, 1, 0)$. Then find the Taylor polynomial of degree 2 of the function $z = z(x, y)$ centered at the point $(2, 1)$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 69. Show that there is a polynomial $P(x, y, z)$ of order 4 such that the set $P(x, y, z) = 0$ is a torus. Show that the gradient of P is nonzero at every point of the torus and conclude that the torus is locally a graph of a smooth function of two variables.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 70. Show that there is no polynomial $P(x, y, z)$ of order less than 4 such that the set $P(x, y, z) = 0$ is a torus.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 71. The cylinder $(x - 1)^2 + y^2 = 1$ intersects with the sphere $x^2 + y^2 + z^2 = 4$ along a curve. This curve has a self-intersection (the curve looks like "8"). Find the angle at which the curve intersects with itself.

Proof. WRITE YOUR SOLUTION HERE.

□