

Homework 5 for Math 1540

Due day: never

Problem 72. Suppose $f : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ is a continuous function defined on

$$\mathbb{R}_+^2 = \{(x, y) : x \in \mathbb{R}, y > 0\}.$$

Assume also that the limits

$$g(u, v) = \lim_{t \rightarrow 0} \frac{f((u+t) \cos v, (u+t) \sin v) - f(u \cos v, u \sin v)}{t},$$

and

$$h(u, v) = \lim_{t \rightarrow 0} \frac{f(u \cos(v+t), u \sin(v+t)) - f(u \cos v, u \sin v)}{t},$$

exist and define continuous functions g, h on the domain

$$D = \{(u, v) : u > 0, 0 < v < \pi\}.$$

Prove that the function f is differentiable on $\mathbb{R}_+^2$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 73. Let $f \in C^1(\mathbb{R})$ be a continuously differentiable function such that $|f'(x)| \leq 1/2$ for all $x \in \mathbb{R}$. Define $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$g(x, y) = (x + f(y), y + f(x)).$$

Prove that

- (1) g is a diffeomorphism,
- (2) $g(\mathbb{R}^2) = \mathbb{R}^2$,
- (3) the area $|g([0, 1]^2)|$ of the image of the unit square belongs to the interval $[3/4, 5/4]$.

Hint: Among other tools use the contraction principle.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 74. Prove that the tangent planes to the surface S defined by $x^2 + y^2 - z^2 = 1$ at the points $(x, y, 0) \in S$ are parallel to the z -axis.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 75. Prove that the stereographic projection π_N from the north pole satisfies

(a)

$$\pi_N(x, y, z) = (u(x, y), v(x, y)) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right),$$

(b)

$$\pi_N^{-1}(u, v) = \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right),$$

(c) $\pi_N^{-1} : \mathbb{R}^2 \rightarrow S^2 \setminus \{N\}$ is a regular parametrization, that it is an immersion and a homeomorphism.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 76. Prove that the tangent space to $O(3)$ at the identity matrix $I \in O(3)$ has the following description:

$$T_I O(3) = \{A \in \mathbb{R}^{3 \times 3} : A + A^T = 0\}.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 77. If f is continuous, show that

$$\int_0^x \int_0^y \int_0^z f(t) dt dz dy = \frac{1}{2} \int_0^x (x-t)^2 f(t) dt.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 78. Prove that $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ embedded as the plane generated by the first n coordinates has $((n+1)$ -dimensional) measure zero.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 79. Prove that the standard ternary Cantor set has measure zero.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 80. Prove that if $f : [0, 1] \rightarrow [0, 1]$ is a continuous function then its graph as a subset of $\mathbb{R}^2$ has measure zero.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 81. Prove that if $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a Lipschitz function, i.e. $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in \mathbb{R}^n$ and some $L > 0$ and $E \subset \mathbb{R}^n$ is a set of measure zero, then $f(E) \subset \mathbb{R}^n$ has measure zero.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 82. Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a continuous function. Let $K \subset \mathbb{R}^3$ be a compact set such that $|f(x) - f(y)| \leq 2012|x - y|^2$ for all $x, y \in K$. Prove that the set $f(K) \subset \mathbb{R}^2$ has measure zero as a subset of $\mathbb{R}^2$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 83. Assume that $f : [0, 1] \rightarrow [0, 1]$ is a continuous function such that the set $\{x \in [0, 1] : f(x) = 1\}$ has measure zero. Prove directly (without using any results like monotone or dominated convergence theorem) that

$$\lim_{n \rightarrow \infty} \int_0^1 f(x)^n dx = 0.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 84. Let $f : [0, 1] \rightarrow (0, \infty)$ be a C^1 function. Find a diffeomorphism of $(0, 1) \times (0, 1)$ onto the domain $\{(x, y) : x \in (0, 1), 0 < y < f(x)\}$. Then apply the change of variables formula to this diffeomorphism to show that the area under the graph of f equals $\int_0^1 f(x) dx$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 85. The mapping $\Phi : \mathbb{R}^2 \rightarrow (0, \infty) \times (-\infty, 0)$, $\Phi(x, y) = (e^x, -e^y)$ is a diffeomorphism (take it for granted; do not prove it). Let $g : [1, e] \times [-e, -1] \rightarrow \mathbb{R}$ be continuous. Express the integral

$$\int_1^e \int_{-e}^{-1} g(x, y) dx dy$$

using Φ as a change of variables (Just write the integral. It is impossible to evaluate it, because we do not know the function g).

Proof. WRITE YOUR SOLUTION HERE. □

Problem 86.

(a) Let $\varphi_i = \varphi_i(x_1, x_2, \dots, x_i) : \mathbb{R}^i \rightarrow \mathbb{R}$ be C^1 functions for $i = 1, 2, \dots, n-1$. Prove that the mapping $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by

$$\Phi(x_1, x_2, \dots, x_n) = (x_1, \varphi_1(x_1) + 2x_2, \varphi_2(x_1, x_2) + 3x_3, \dots, \varphi_{n-1}(x_1, x_2, \dots, x_{n-1}) + nx_n)$$

is a diffeomorphism onto an open subset of $\mathbb{R}^n$.

(b) Find the volume of $\Phi((0, 1)^n)$, where $(0, 1)^n$ is an open unit cube in $\mathbb{R}^n$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 87. Let $\gamma : \mathbb{R} \rightarrow \mathbb{R}^n$ and $\mathbf{v}_i : \mathbb{R} \rightarrow \mathbb{R}^n$, $i = 1, 2, \dots, n-1$, be C^∞ smooth functions such that for any $t \in \mathbb{R}$ the vectors

$$\gamma'(t), \mathbf{v}_1(t), \dots, \mathbf{v}_{n-1}(t)$$

form an orthonormal basis of $\mathbb{R}^n$ (here we differentiate γ but **do not** differentiate $\mathbf{v}_i$, $i = 1, 2, \dots, n-1$).

Consider the mapping $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by

$$\Phi(x_1, \dots, x_n) = \gamma(x_n) + \sum_{i=1}^{n-1} x_i \mathbf{v}_i(x_n).$$

(a) Find the derivative $D\Phi(x_1, \dots, x_n)$;

(b) Prove that Φ is a diffeomorphism in a neighborhood of and point of the form $(0, \dots, 0, x_n)$;

(c) Find the limit

$$\lim_{r \rightarrow 0} \frac{|\Phi(B^n(0, r))|}{|B^n(0, r)|},$$

where $B^n(0, r)$ denotes the ball of radius r centered at the origin and $|A|$ stands for the volume of the set A .

Proof. WRITE YOUR SOLUTION HERE. □

Problem 88. Let $\Phi : \mathbb{R}^2 \rightarrow \Phi(\mathbb{R}^2) \subset \mathbb{R}^2$ be a diffeomorphism. Prove that

$$\int_{B^2(0,1)} \|D\Phi\| = \int_{\Phi(B^2(0,1))} \|D(\Phi^{-1})\|,$$

where $\|A\| = (\sum_{i,j=1}^2 a_{ij}^2)^{1/2}$ is the Hilbert-Schmidt norm of the matrix.

Hint. Compare $\|A\|$ and $\|A^{-1}\|$ for a 2×2 matrix.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 89. Prove that if $K \in C^1(\mathbb{R}^2 \setminus \{(0, 0)\})$ satisfies the estimate

$$|\nabla K(x)| \leq \frac{1}{|x|^3} \quad \text{for all } x \neq (0, 0)$$

then there is a constant $C > 0$ such that

$$\iint_{\{x \in \mathbb{R}^2: |x| > 2|y|\}} |K(x - y) - K(x)| dx \leq C$$

for all $y \in \mathbb{R}^2$.

Hint: Use the mean value theorem to estimate $|K(x - y) - K(x)|$ and then integrate in polar coordinates.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 90. Prove that

$$\int_0^1 \int_0^1 \frac{1}{1 - xy} dx dy = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

The integral is understood as an improper integral $\lim_{t \rightarrow 1^-} \int_0^t \int_0^t \dots$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 91. Suppose that $f : \mathbb{R}^2 \rightarrow [0, \infty)$ is uniformly continuous on $\mathbb{R}^2$ and

$$\sup_{r>0} \iint_{\{x^2+y^2 \leq r^2\}} f(x, y) dx dy < \infty.$$

Prove that $\lim_{|(x,y)| \rightarrow \infty} f(x, y) = 0$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 92. Use Green's theorem to prove the following result: If the vertices of a polygon, in counterclockwise order, are $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$, then the area of the polygon is

$$A = \frac{1}{2} \sum_{i=1}^n (x_i y_{i+1} - x_{i+1} y_i),$$

where we use notation $x_{n+1} = x_1, y_{n+1} = y_1$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 93. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with C^1 boundary and let $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $\Phi(x, y) = (u(x, y), v(x, y))$, be a C^2 diffeomorphism. Prove that

$$\int_{\partial\Omega} uv_x dx + uv_y dy = \pm |\Phi(\Omega)|,$$

where $|\Phi(\Omega)|$ denotes the area of $\Phi(\Omega)$ and $\partial\Omega$ has positive orientation. Show on examples that both cases $+\lvert\Phi(\Omega)\rvert$ and $-\lvert\Phi(\Omega)\rvert$ are possible.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 94. Suppose that $f \in C^2(\mathbb{R}^3)$ is constant in a neighborhood of the boundary of a ball $B \subset \mathbb{R}^3$. Prove that

$$\iiint_B (f_{xx} + f_{yy} + f_{zz}) dV = 0.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 95. Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and $f \in C^\infty(\mathbb{R}^2)$. Suppose that $f(x, y) = 1$ for all $(x, y) \in \partial D$. Prove that

$$\iint_D \left(2f(x, y) + x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} \right) dA = 2\pi.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 96. Let $x = (x_1, x_2)$ and $f(x)$ be a C^2 function on $\mathbb{R}^2$ such that $|\partial^2 f(x)/\partial x_j \partial x_k| \leq 1$ for all $x \in \mathbb{R}^2$ and $j, k \in \{1, 2\}$. Prove that

$$\left| \iint_{|x| \leq 1} f(x) dA - \pi f(0, 0) \right| \leq 3/2.$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 97. Suppose $f(x, y)$ is a C^2 function on the unit disk $\mathbb{D} = \{(x, y) \mid x^2 + y^2 \leq 1\}$ such that for some $M > 0$ and all $(x, y) \in \mathbb{D}$,

$$[f_{xx}(x, y)]^2 + 2[f_{xy}(x, y)]^2 + [f_{yy}(x, y)]^2 \leq M.$$

If $f(0, 0) = f_x(0, 0) = f_y(0, 0) = 0$, show that

$$\left| \iint_{\mathbb{D}} f(x, y) dx dy \right| \leq \frac{\pi \sqrt{M}}{4}$$

Proof. WRITE YOUR SOLUTION HERE. □

Problem 98. Let f be a polynomial of total degree at most three in $(x, y, z) \in \mathbb{R}^3$. Prove that:

$$\int_{x^2+y^2+z^2 \leq 1} f(x, y, z) dx dy dz = \frac{4\pi f((0, 0, 0))}{3} + \frac{2\pi (\Delta f)((0, 0, 0))}{15}.$$

Here $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian operator on $\mathbb{R}^3$.

Proof. WRITE YOUR SOLUTION HERE. □

Problem 99. Suppose that $K \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ and

$$K(x) = \frac{K(x/\|x\|)}{\|x\|} \quad \text{for all } x \in \mathbb{R}^2 \setminus \{0\}.$$

- (a) Prove that $\nabla K(tx) = t^{-2} \nabla K(x)$ for $x \neq 0$ and $t > 0$.
- (b) Use the divergence theorem to prove that (on both sides we integrate vector valued functions)

$$\int_{\{1 \leq \|x\| \leq 2019\}} \nabla K(x) dx = \int_{\partial\{1 \leq \|x\| \leq 2019\}} K(x) \vec{n} d\sigma(x).$$

(c) Prove that

$$\int_{\{1 \leq \|x\| \leq 2019\}} \nabla K(x) dx = 0.$$

Hint. Show first that $K(tx) = t^{-1}K(x)$ for $x \neq 0$, $t > 0$. In (a) differentiate $K(tx)$. Part (a) is not needed for parts (b) and (c).

Proof. WRITE YOUR SOLUTION HERE. □

Problem 100. For $x = (x_1, x_2) \in \mathbb{R}^2$, let $|x| = \sqrt{x_1^2 + x_2^2}$. Let $D = \{x \in \mathbb{R}^2 : |x| < 1\}$ and let $f : \overline{D} \rightarrow \mathbb{R}$ be continuous on $\overline{D}$. Prove that

$$\lim_{n \rightarrow \infty} \iint_D (n+2)|x|^n f(x) dA = \int_0^{2\pi} f(\cos t, \sin t) dt.$$

Proof. WRITE YOUR SOLUTION HERE. □