Homework 2 for Math 1540
Due day: February 14, Canvas.

Problem 11. Let $(x_n)_{n=1}^{\infty}$ be a sequence of points in $\mathbb{R}^3$ such that $|x_{n+1} - x_n| \leq 1/(n^2+n)$, $n \geq 1$. Show that (x_n) converges.

Proof. □

Problem 12. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a norm, that is for all $x, y \in \mathbb{R}^n$ and all $t \in \mathbb{R}$ we have

(1) $F(x) \geq 0$ and $F(x) = 0$ if and only if $x = 0$,
(2) $F(x + y) \leq F(x) + F(y)$,
(3) $F(tx) = |t|F(x)$.

Prove the following statements:

(a) F is bounded on the unit sphere.
(b) F is continuous.
(c) There are constants $A, B > 0$ such that $A\|x\| \leq F(x) \leq B\|x\|$, for all $x \in \mathbb{R}^n$.

Hint. *It is useful to represent $v \in \mathbb{R}^n$ in a standard basis. For part (c) think what happens on the unit sphere.*

Proof. □

Problem 13. Prove that $(\mathbb{R}^n, \varrho)$, where

$$\varrho(x, y) = \frac{|x - y|}{1 + |x - y|}$$

is a metric space.

Proof. □

Problem 14. For what values of $\alpha > 0$, $(\mathbb{R}^n, \varrho_\alpha)$, is a metric space, where

$$\varrho_\alpha(x, y) = |x - y|^\alpha.$$

Proof. □

Problem 15. Prove that if $A \subset X$ is a subset of a separable metric space, then (A, d) is separable.

Proof. □

Problem 16. Prove that if $E_2 \subset E_2 \subset E_3 \subset \ldots$ is a decreasing sequence of non-empty closed sets in a complete metric space and $\operatorname{diam} E_n = \sup_{x,y\in E_n} d(x, y) \to 0$ as $n \to \infty$, then $\bigcap_{n=1}^{\infty} E_n \neq \varnothing$.

Proof. □

Problem 17. Let $f_n : X \to \mathbb{R}$, $n = 1, 2, \ldots$ be a sequence of continuous functions on a metric space X such that the series $\sum_{n=1}^{\infty} f_n(x)$ converges for all $x \in X$ and

$$\sup_{x\in X} \left(\sum_{n=1}^{\infty} f_n(x)^2 \right)^{1/2} < \infty.$$

Prove that if a sequence of real numbers c_n, $n = 1, 2, \ldots$ satisfies $\sum_{n=1}^{\infty} c_n^2 < \infty$, then the series

$$\sum_{n=1}^{\infty} c_n f_n(x)$$

converges everywhere to a continuous function.

Proof. □

Problem 18. Prove that if $x \in \ell^2$ and $r > 0$, then the ball $\bar{B}(x, r)$ in ℓ^2 is not compact.

Proof. □

Problem 19. Prove that ℓ^2 is separable.

Proof. □

Problem 20. Provide a simple example of a metric space that is complete and not separable.

Proof. □