Homework 3 for Math 1540
Due day: February 21, Canvas.

Problem 21. We say that a subset E of a metric space is G_δ if there is a sequence of open sets $\{U_i\}_{i=1}^\infty$ such that $E = \bigcap_{i=1}^\infty U_i$. Let $f : X \to \mathbb{R}$ be a function defined on a metric space. Prove that the set

$$\{x \in X : f \text{ is continuous at } x\}$$

is G_δ.
Hint: *Let m and n denote positive integers. It is very easy to verify and you can take it for granted that f is continuous at x iff*

$$\forall n\, \exists m\, \forall z, w \left(d(x,z) < m^{-1},\, d(x,w) < m^{-1} \implies |f(z) - f(w)| < n^{-1} \right) .$$

Now the idea is to define sets $U_{n,m}$ and then characterize the set of all points of continuity of f in terms of $U_{n,m}$.

Proof. □

Problem 22. Prove (using only the material covered in the course) that there is no continuous and one-to-one function $f : \mathbb{R}^2 \to \mathbb{R}$. **Hint:** *Assume that such a function exists and then restrict the function to the unit circle in $\mathbb{R}^2$.*

Proof. □

Problem 23. Let (X, d) be a compact metric space and $z \in X$. Let $T : X \to X$ be a function which satisfies $d(x, y) \leq d(T(x), T(y))$ for all $x, y \in X$, i.e. the distances are non-decreasing under the mapping T. Define $\{x_n\}$ by

$$x_1 = T(z) \quad \text{and } x_{n+1} = T(x_n) \text{ for } n \geq 1.$$

Prove that there exists a subsequence of $\{x_n\}$ which converges to z.

Proof. □

Problem 24. Let (X, d) be a nonempty complete metric space. Let $S : X \to X$ be a given mapping and write S^2 for $S \circ S$ i.e. $S^2(x) = S(S(x))$. Suppose that S^2 is a contraction. Show that S has a unique fixed point.

Proof. □

Problem 25. Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix.

(a) Show that there exists $\Lambda > 0$ such that $|x| \leq \Lambda |Ax|$ for all $x \in \mathbb{R}^n$.
(b) Let $0 \leq \lambda < \Lambda^{-1}$ and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz map with $|f(x) - f(y)| \leq \lambda |x - y|$ for all $x, y \in \mathbb{R}^n$. Show that for any $y \in \mathbb{R}^n$, there exists exactly one $x \in \mathbb{R}^n$ satisfying

$$Ax + f(x) = y.$$

Proof. □

Problem 26. Prove that is a sequence of L-Lipschitz functions $f_k : \mathbb{R}^n \to \mathbb{R}$ is such that $(f_k(q))_k$ convergent for every $q \in \mathbb{R}^n$ with all rational coordinates, then the sequence $(f_k(x))_k$ is convergent for every $x \in \mathbb{R}^n$.

Proof. □

Problem 27. Let $\{f_n\}_{n=1}^\infty$ be a uniformly bounded equicontinuous sequence of real-valued functions on a compact metric space (X, d).

(a) Prove that the family $g_n(x) = \max\{f_1(x), \ldots, f_n(x)\}$, $n \in \ltimes$ is uniformly bounded and equicontinuous.

(b) Prove that the sequence $\{g_n\}_{n=1}^{\infty}$ converges uniformly on X.

Proof. □

Problem 28. Show that

$$K := \{f \in C^1((0,1)) \cap C^0([0,1]) : \quad f'(x) = |f(x)| \text{ and } |f(x)| \leq 2 \quad \text{holds for all } x \in (0,1)\}$$

is a compact set when equipped with the metric

$$d(f,g) := \sup_{x\in[0,1]} |f(x) - g(x)| + \sup_{x\in(0,1)} |f'(x) - g'(x)|.$$

Proof. □

Problem 29. Let $h : [0, 2\pi] \to \mathbb{R}$ be continuous and $D = \{(x,y) : \quad x^2 + y^2 \leq 1\}$. Define the sequence of functions $\{F_n\}$ on D by

$$F_n(x,y) = \int_0^{2\pi} \cos(x \sin\theta + y\cos\theta - nh(\theta))d\theta$$

for $(x,y) \in D$ and $n \in \mathbb{N}$. Prove that $\{F_n\}$ has a subsequence that converges uniformly on D.

Proof. □

Problem 30. Let $f : \mathbb{R}^n \to \mathbb{R}$ be bounded and uniformly continuous. Prove that the family of functions $\{g_z\}_{z\in\mathbb{R}^n}$, $g_z(x) = f(x)f(x-z)$ is equicontinuous.

Proof. □