Homework 4 for Math 1540
Due day: February 28, Canvas.

Problem 31. If $f = (f_1, \ldots, f_n) : [a, b] \to \mathbb{R}^n$ is a continuous function, then we define

$$\int_a^b f(t)\,dt = \left\langle \int_a^b f_1(t)\,dt, \ldots, \int_a^b f_n(t)\,dt \right\rangle .$$

Prove that

$$\left\| \int_a^b f(t)\,dt \right\| \leq \int_a^b \|f(t)\|\,dt.$$

Proof. □

Problem 32. Let the functions $f_n : [0, 1] \to [0, 1]$, $n = 1, 2, \ldots$, satisfy $|f_n(x) - f_n(y)| \leq |x - y|$ whenever $|x - y| \geq 1/n$. Prove that the sequence $\{f_n\}_{n=1}^{\infty}$ has a uniformly convergent subsequence.

Proof. □

Problem 33. We know that every continuous function $f : [a, b] \to \mathbb{R}$ can be uniformly approximated by polynomials (Weierstrass' theorem). Prove that if a continuous function $f : \mathbb{R} \to \mathbb{R}$ can be uniformly approximated by polynomials on all of $\mathbb{R}$, then f is a polynomial.

Proof. □

Problem 34. We know that if $f : [-1, 1] \to \mathbb{R}$ is continuous and

$$\int_{-1}^{1} f(x)x^n\,dx = 0 \tag{1}$$

for $n = 0, 1, 2, 3, \ldots$, then $f(x) = 0$ for all $-1 \leq x \leq 1$. We proved it using the Weierstrass theorem. Suppose now that $f : [-1, 1] \to \mathbb{R}$ is continuous and (1) holds for all $n \geq 2024$. Prove that $f(x) = 0$ for all $-1 \leq x \leq 1$?

Proof. □

Problem 35. Prove that if $f : [0, 1] \to \mathbb{R}$ is such that

$$\int_0^1 f(x)e^{nx}\,dx = 0 \quad \text{for all } n = 0, 1, 2, \ldots,$$

then $f(x) = 0$ for all $0 \leq x \leq 1$. Provide two proofs following the methods:

(a) Use the Stone-Weierstrass theorem.
(b) Use the change of variables formula and apply the Weierstrass theorem.

Proof. □

Problem 36. Prove that the trigonometric polynomials

$$T(x) = \sum_{k=0}^{n} a_k \cos kx + \sum_{k=0}^{n} b_k \sin kx, \quad a_k, b_k \in \mathbb{R}$$

form an algebra. *Hint:* $\cos x + i \sin x = e^{ix}$.

Proof. □

Problem 37. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in the complex plane. Let $\mathcal{A}$ be the algebra of functions of the form

$$f(e^{i\theta}) = \sum_{n=0}^{N} c_n e^{in\theta}, \quad c_n \in \mathbb{C},\, \theta \in \mathbb{R}.$$

It is easy to see that $f \equiv 1$ belongs of $\mathcal{A}$ and $\mathcal{A}$ separates points (do not prove it). Prove that there are complex valued functions on S^1 that cannot be uniformly approximated by functions in $\mathcal{A}$. *Hint:* For $f \in \mathcal{A}$

$$\int_0^{2\pi} f(e^{i\theta}) e^{i\theta}\, d\theta = 0\,.$$

Proof. □

Problem 38. Prove that complex polynomials

$$p(z) = \sum_{n=0}^{N} c_n z^n, \quad c_n \in \mathbb{C}$$

are not dense in $C(\overline{D}, \mathbb{C})$, where

$$\overline{D} = \{z \in \mathbb{C} : |z| \leq 1\}$$

is the unit disc in $\mathbb{C}$. *Hint:* Consider $f(z) = \overline{z}$. Is the previous exercise helpful?

Proof. □