Homework 5 for Math 1540
Due day: March 18, Canvas.

Problem 39. Let $A = [a_{ij}]$ be the matrix of a linear mapping $A \in L(\mathbb{R}^n, \mathbb{R}^m)$. Prove that the norm

$$\|A\| = \sup_{\|x\|=1} \|Ax\|$$

satisfies the inequality

$$\|A\| \leq \left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2 \right)^{1/2}$$

Hint: You may use the following argument: Write the components of the vector Ax as scalar products of rows on A and x. Then use the Schwarz inequality to estimate the length of the vector Ax.

Proof. □

Problem 40. Prove directly from the definition (do not use the Sylvester theorem) that the matrix

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

is positive definite iff $a > 0$ and $ad - b^2 > 0$

Proof. □

Problem 41. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by $F(x, y) = f(xy)$. Prove that

$$x\frac{\partial F}{\partial x} = y\frac{\partial F}{\partial y}\,.$$

Proof. □

Problem 42. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree m if $f(tx) = t^m f(x)$ for all $x \in \mathbb{R}^n$ and all $t > 0$. Prove that if f is differentiable on $\mathbb{R}^n$ and homogeneous of degree m, then

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(x) = mf(x) \quad \text{for all } x \in \mathbb{R}^n.$$

Proof. □

Problem 43. We know that a function $f(x, y)$ is differentiable at $(0, 0)$. We also know the directional derivatives

$$D_u f(0,0) = 1 \quad \text{where } u = [1/\sqrt{5}, 2/\sqrt{5}],$$
$$D_v f(0,0) = 1 \quad \text{where } v = [1/\sqrt{2}, 1/\sqrt{2}].$$

Find the gradient $\nabla f(0, 0)$.

Proof. □

Problem 44. Let $f \in C^1(\mathbb{R}^2)$ be such that $f(1, 1) = 1$ and $\nabla f(1, 1) = (a, b)$. Let $\varphi(x) = f(x, f(x, f(x, x)))$. Find $\varphi(1)$ and $\varphi'(1)$.

Proof. □

Problem 45. A function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable. Find the derivative of the function

$$F(t) = (f(t, t^2, \ldots, t^n))^2, \quad t \in \mathbb{R}$$

of one variable.

Proof. □

Problem 46. Verify by a direct computation that the vector field $F(x) = x|x|^{-n}$ defined on $\mathbb{R}^n \setminus \{0\}$ is divergence free, i.e.

$$\operatorname{div} F(x) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{x_i}{|x|^n}\right) = 0 \quad \text{for all } x \neq 0.$$

Proof. □

Problem 47. Prove that for $\alpha > 0$ the function $\Phi : \mathbb{R}^n \to \mathbb{R}^n$,

$$\Phi(x) = x|x|^{\alpha}$$

is of class C^1. Find $D\Phi(x)$.

Proof. □

Problem 48. Find all the points $(x, y) \in \mathbb{R}^2$ where the function

$$f(x, y) = |e^x - e^y| \cdot (x + y - 2)$$

is differentiable.

Proof. □

Problem 49. Consider the function $g : \mathbb{R}^2 \to \mathbb{R}$ given by

$$g(x, y) = x^{2/3}y^{2/3}, \text{ for all } (x, y) \in \mathbb{R}^2 .$$

Prove that g is *differentiable* at $(0, 0)$.

Proof. □

Problem 50. Prove that is the partial derivatives (of first order) of a function $f : \mathbb{R}^n \to \mathbb{R}$ exist everywhere and they are bounded, then the function f is continuous.

Proof. □

Problem 51. Prove that if $f, g \in C^k(\Omega)$, $\Omega \subset \mathbb{R}^n$, then for any multiindex α with $|\alpha| \leq k$ we have

$$D^{\alpha}(fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^{\beta} f D^{\alpha-\beta} g \,,$$

where $\beta \leq \alpha$ means that $\beta_i \leq \alpha_i$ for $i = 1, 2, \ldots, n$, $\alpha - \beta = (\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n)$ and

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha - \beta)!} \,.$$

Proof. □

Problem 52. Let $f \in C^2(\mathbb{R}^2)$. Suppose that $\nabla f = 0$ on a compact set $E \subset \mathbb{R}^2$. Prove that there is a constant $M > 0$ such that $|f(x) - f(y)| \leq M|x - y|^2$ for all $x, y \in E$.

Proof. □

Problem 53.[A smooth function with compact support]
Consider an open ball $B = B(a, r) \subset \mathbb{R}^n$. Prove that the function

$$\varphi(x) = \begin{cases} \exp\left(\frac{1}{|x-a|^2 - r^2}\right) & \text{if } x \in B \,, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus B \,, \end{cases}$$

in infinitely differentiable on $\mathbb{R}^n$.

Proof. □