Homework 6 for Math 1540
Due day: April 2, Canvas.

Problem 54. Show that if $f \in C^2(\mathbb{R}^n)$ has a local minimum at $x = 0$ and $f(0) = 0$, then

$$f(x) = \int_0^1 (1-t) x^T D^2 f(tx)\, x\, dt$$

(show all details of the derivation of the above formula). Is the above formula is still true if $f(0) = 0$, but instead of a local minimum we have a local maximum?

Proof. □

Problem 55. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{x^2(y^4+2x)}{x^2+y^4} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Prove that f is differentiable at $(0,0)$.

Proof. □

Problem 56. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Prove that the mixed partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ exist everywhere in $\mathbb{R}^2$, but

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0).$$

Proof. □

Problem 57. Prove that the function

$$f(x) = \begin{cases} \frac{x|y|}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

has all directional derivatives $D_v f(0,0)$ at the origin, but f is not differentiable at $(0,0)$.

Proof. □

Problem 58. Let $Q : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ be a linear mapping such that $Qf \geq 0$ whenever $f \in C^\infty(\mathbb{R}^n)$ satisfies $f(0) = 0$ and $f(x) \geq 0$ in a neighborhood of 0. Prove that there are real numbers a_{ij}, b_i and c such that

$$Qf = \sum_{i,j=1}^n a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(0) + \sum_{i=1}^n b_i \frac{\partial f}{\partial x_i}(0) + cf(0) \quad \text{for all } f \in C^\infty(\mathbb{R}^n).$$

Proof. □

Problem 59. Let $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$, where $\Omega \subset \mathbb{R}^n$ is open and bounded. Let $\Delta f = \sum_{i=1}^n \partial^2 f / \partial x_i^2$ be the Laplace operator.

(a) Show that if for some $\varepsilon > 0$ and $x_0 \in \Omega$ we have $\Delta f(x_0) \geq \varepsilon$, then f has no local maximum at x_0.
(b) Conclude that if $\Delta f(x) \geq \varepsilon$ for some $\varepsilon > 0$ and all $x \in \Omega$, then we have $\sup_\Omega f = \sup_{\partial\Omega} f$.
(c) Conclude that if $\Delta f(x) \geq 0$ for all $x \in \Omega$, then we have $\sup_\Omega f = \sup_{\partial\Omega} f$.

Hint for part (c): *Observe that* $\Delta|x|^2 = 2n$*. Use it to modify a function* f *in (c) so that you can apply part (b).*

Proof. □

Definition. $f : \mathbb{R}^n \to \mathbb{R}$ is convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \quad \text{for all } x, y \in \mathbb{R}^n \text{ and } \lambda \in [0,1].$$

Problem 60. Prove that if $f : \mathbb{R}^n \to \mathbb{R}$ is convex, then

$$f\Big(\sum_{i=1}^k \lambda_i x_i\Big) \leq \sum_{i=1}^k \lambda_i f(x_i) \text{ whenever } x_1, \dots, x_k \in \mathbb{R}^n,\ \lambda_i \geq 0,\ \sum_{i=1}^k \lambda_i = 1.$$

Proof. □

Problem 61. Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. Prove that if partial derivatives

$$\frac{\partial f}{\partial x_i}(x_0), \quad i = 1, 2, \dots, n,$$

exist, then f is differentiable at x_0.

Proof. □

Problem 62. Let $E \subset \mathbb{R}^n$ be closed. Prove that $\forall x \in \mathbb{R}^n\ \exists y \in E\ \operatorname{dist}(x, E) = |x - y|$. Show an example of E and x such that there are infinitely many points y satisfying $\operatorname{dist}(x, E) = |x - y|$.

Proof. □

Problem 63. Let $E \subset \mathbb{R}^n$ be closed. Assume that the function $x \mapsto \operatorname{dist}(x, E)$ is differentiable at $x_o \in \mathbb{R}^n$. Prove that there is exactly one $y \in E$ such that $\operatorname{dist}(x_o, E) = |x_o - y|$. **Hint:** *Assume we have two such points* y_1 *and* y_2*. Investigate directional derivatives in the directions of* y_1 *and* y_2*.*

Proof. □

Problem 64. If $W \subset \mathbb{R}^n$ is convex and closed, then $\forall x \in \mathbb{R}^n\ \exists!\, y \in E\ \operatorname{dist}(x, E) = |x - y|$. This is obvious. Don't prove it. Denote this unique y by $\pi_W(x)$ so we have a function $\pi_W : \mathbb{R}^n \to W$. Prove that π_W is 1-Lipschitz i.e., $|\pi_W(x) - \pi_W(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}^n$.

Proof. □

Problem 65. Let $W \subset \mathbb{R}^n$ be convex and closed and let $f(x) = \operatorname{dist}(x, W)^2 = |x - \pi_W(x)|^2$. Prove that $f \in C^1(\mathbb{R}^n)$ and that $Df(x) = 2(x - \pi_W(x))$ for all $x \in \mathbb{R}^n$. Conclude that the gradient of the function f is Lipschitz continuous.

Proof. □