

Homework 1 for Math 1540

Due Sunday, January 25, 2026 (via Canvas).

Problem 1. Prove that in any metric space $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$ and $\text{cl}(A \cap B) \subset \text{cl}(A) \cap \text{cl}(B)$. Provide an example showing that $\text{cl}(A \cap B) \neq \text{cl}(A) \cap \text{cl}(B)$.

Proof.

□

Problem 2. Show an example of a metric space such that the a closed ball $\bar{B}(x, r) = \{y : d(x, y) \leq r\}$, $r > 0$, is *not* equal to the closure of the ball $B(x, r) = \{y : d(x, y) < r\}$, $\bar{B}(x, r) \neq \text{cl}(B(x, r))$.

Proof.

□

Problem 3. Prove that if (X, d) is separable and $A \subset X$, then (A, d) is separable.

Proof.

□

Problem 4. Prove that the Hilbert space ℓ^2 is separable.

Proof.

□

Problem 5. Prove that ℓ^∞ is not separable.

Proof.

□

Problem 6. Prove that if X is compact, $T : X \rightarrow X$ is continuous and $d(T(x), T(y)) < d(x, y)$ whenever $x \neq y$, then T has a fixed point i.e., there is x such that $T(x) = x$.

Proof.

□

Problem 7. Let (X, d) be a compact metric space and let $f : X \rightarrow X$ be an isometry i.e., $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. Prove that f is onto i.e., $f(X) = X$.

Proof.

□

Problem 8. Let $E \subset \mathbb{R}^n$ be compact and let

$$K = \{x \in \mathbb{R}^n : |x - y| = 1 \text{ for some } y \in E\}.$$

Prove that K is compact.

Proof.

□

Problem 9. Recall that a homeomorphism $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a one-to-one and onto map such that both f and f^{-1} are continuous. Prove that if $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a homeomorphism, then $\lim_{|x| \rightarrow \infty} |f(x)| = \infty$.

Proof.

□

Problem 10. Let $f_n : X \rightarrow \mathbb{R}$, $n = 1, 2, \dots$ be a sequence of continuous functions on a metric space X such that the series $\sum_{n=1}^{\infty} f_n(x)$ converges for all $x \in X$ and

$$\sup_{x \in X} \left(\sum_{n=1}^{\infty} f_n(x)^2 \right)^{1/2} < \infty.$$

Prove that if a sequence of real numbers c_n , $n = 1, 2, \dots$ satisfies $\sum_{n=1}^{\infty} c_n^2 < \infty$, then the series

$$\sum_{n=1}^{\infty} c_n f_n(x)$$

converges everywhere to a continuous function.

Proof.

□

Problem 11. Prove that the set of irrational numbers is G_δ .

Proof.

□

Problem 12. Prove directly from the Baire theorem that the set of rational numbers is not G_δ .

Proof.

□

Problem 13. Let $E \subset \mathbb{R}$ be a G_δ set. Prove that there is a function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that the set of points where f is continuous equals E .

Proof.

□

Problem 14. Prove that if $f : X \rightarrow Y$ is continuous, then for any $M > 0$ the set

$$\{x \in X : \operatorname{osc} f(x) < M\} \text{ is open.}$$

Proof.

□

Problem 15. Using the fact that there is a function $f \in C([0, 1])$ that is nowhere differentiable, prove directly that the set of continuous nowhere differentiable functions is dense in $C([0, 1])$.

Proof.

□