

Homework 2 for Math 1540

Due Sunday, February 1, 2026 (via Canvas).

Problem 16. Prove that $(\mathbb{R}^n, \varrho)$, where

$$\varrho(x, y) = \frac{|x - y|}{1 + |x - y|}$$

is a metric space.

Proof. □

Problem 17. Prove that if (X, d) is a metric space and $0 < \alpha \leq 1$, then (X, d^α) is a metric space, where $d^\alpha(x, y) = (d(x, y))^\alpha$ for all $x, y \in X$.

Proof. □

Problem 18. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}$ be a norm, that is for all $x, y \in \mathbb{R}^n$ and all $t \in \mathbb{R}$ we have

- (1) $F(x) \geq 0$ and $F(x) = 0$ if and only if $x = 0$,
- (2) $F(x + y) \leq F(x) + F(y)$,
- (3) $F(tx) = |t|F(x)$.

Prove the following statements:

- (a) F is bounded on the unit sphere.
- (b) F is continuous.
- (c) There are constants $A, B > 0$ such that $A\|x\| \leq F(x) \leq B\|x\|$, for all $x \in \mathbb{R}^n$, where $\|x\|$ is the Euclidean norm of the vector.

Hint. It is useful to represent $v \in \mathbb{R}^n$ in a standard basis. For part (c) think what happens on the unit sphere.

Proof. □

Problem 19. Let $f, g : X \rightarrow \mathbb{R}$ be bounded uniformly continuous real functions define on the metric space (X, d) . Prove that the product $x \mapsto f(x)g(x)$ is also uniformly continuous. Show also that the conclusion is false when the boundedness hypothesis is omitted.

Proof. □

Problem 20. Prove that if $E_2 \subset E_1 \subset E_3 \subset \dots$ is a decreasing sequence of non-empty closed sets in a complete metric space and $\text{diam } E_n = \sup_{x, y \in E_n} d(x, y) \rightarrow 0$ as $n \rightarrow \infty$, then $\bigcap_{n=1}^{\infty} E_n \neq \emptyset$.

Proof. □

Problem 21. Show that a metric space X is disconnected if and only if there is a proper set $A \subset X$, i.e. $A \neq \emptyset$ and $A \neq X$, such that $\text{bd}(A) = \emptyset$.

Proof. □

Problem 22. Prove that an open set $\Omega \subset \mathbb{R}^n$ is connected if and only if Ω is path connected.

Proof. □

Problem 23. Prove (using only the material covered in the course) that there is no continuous and one-to-one function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. **Hint:** Assume that such a function exists and then restrict the function to the unit circle in $\mathbb{R}^2$.

Proof. □

Problem 24. Let $(x_n)_{n=1}^{\infty}$ be a sequence of points in $\mathbb{R}^3$ such that $|x_{n+1} - x_n| \leq 1/(n^2 + n)$, $n \geq 1$. Show that (x_n) converges.

Proof. □