

Homework 3 for Math 1540

Due Sunday, February 8, 2026 (via Canvas).

Problem 25. Let (X, d) be a metric space. Prove that if $A_i \subset X$, $i \in I$, are connected subsets and $\bigcap_{i \in I} A_i \neq \emptyset$, then the set $\bigcup_{i \in I} A_i$ is connected.

Proof. □

Problem 26. Prove that if (X, d_X) and (Y, d_Y) are connected metric spaces, then their product $(X \times Y, d_{X \times Y})$ is connected.

Proof. □

Problem 27. Prove that there is a unique continuous function $f : [0, 1] \rightarrow \mathbb{R}$ such that

$$f(x) = \sin x + \int_0^1 \frac{f(y)}{e^{x+y+1}} dy \quad \text{for all } x \in [0, 1]$$

Proof. □

Problem 28. Prove that:

(a) There is a unique continuous function $f : [0, 1] \rightarrow \mathbb{R}$ such that

$$f(x) = 1 + \int_0^x t^2 f(t) dt \quad \text{for all } x \in [0, 1];$$

(b) The function from (a) is of class $f \in C^\infty(0, 1)$.

Proof. □

Problem 29. Let X be a complete metric space. Prove that if $S : X \rightarrow X$ is such that $S^2 = S \circ S$ is a contraction, then S has a unique fixed point.

Proof. □

Problem 30. Let (X, d) be a complete metric space, and that $f : X \rightarrow X$ is a function. For any $m \in \mathbb{N}$, let f^n denote the n -times iterated composition of the function f with itself. (It is defined recursively by $f^1 := f$, $f^n := f \circ f^{n-1}$ for $n \geq 2$.) Let

$$\lambda := \liminf_{n \rightarrow \infty} \sup_{x, y \in X, x \neq y} \frac{d(f^n(x), f^n(y))}{d(x, y)}.$$

Prove that if $\lambda < 1$, then f admits a unique fixed point.

Proof. □

Problem 31. Let (X, d) be a compact metric space and $z \in X$. Let $T : X \rightarrow X$ be a function which satisfies $d(x, y) \leq d(T(x), T(y))$ for all $x, y \in X$, i.e. the distances are non-decreasing under the mapping T . Define $\{x_n\}$ by

$$x_1 = T(z) \quad \text{and} \quad x_{n+1} = T(x_n) \quad \text{for } n \geq 1.$$

Prove that there exists a subsequence of $\{x_n\}$ which converges to z .

Proof. □

Problem 32. Let $A \in \mathbb{R}^{n \times n}$ be an invertible matrix.

(a) Show that there exists $\Lambda > 0$ such that $|x| \leq \Lambda |Ax|$ for all $x \in \mathbb{R}^n$.

(b) Let $0 \leq \lambda < \Lambda^{-1}$ and let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a Lipschitz map with $|f(x) - f(y)| \leq \lambda |x - y|$ for all $x, y \in \mathbb{R}^n$. Show that for any $y \in \mathbb{R}^n$, there exists exactly one $x \in \mathbb{R}^n$ satisfying

$$Ax + f(x) = y.$$