

Homework 4 for Math 1540

Due Sunday, February 15, 2026 (via Canvas).

Problem 33. Suppose that a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous. Suppose that there is a sequence of continuous functions $f_k : \mathbb{R}^n \rightarrow \mathbb{R}$, each with compact support, such that $f_k \rightrightarrows f$ uniformly on $\mathbb{R}^n$ as $k \rightarrow \infty$. Prove that $\lim_{|x| \rightarrow \infty} f(x) = 0$.

Proof. □

Problem 34. Prove that if a sequence of L -Lipschitz functions $f_k : \mathbb{R}^n \rightarrow \mathbb{R}$ is such that $(f_k(q))_k$ convergent for every $q \in \mathbb{R}^n$ with all rational coordinates, then the sequence $(f_k(x))_k$ is convergent for every $x \in \mathbb{R}^n$.

Proof. □

Problem 35. Let $\{f_n\}_{n=1}^\infty$ be a uniformly bounded equicontinuous sequence of real-valued functions on a compact metric space (X, d) .

(a) Prove that the family $g_n(x) = \max\{f_1(x), \dots, f_n(x)\}$, $n \in \mathbb{N}$ is uniformly bounded and equicontinuous.

(b) Prove that the sequence $\{g_n\}_{n=1}^\infty$ converges uniformly on X .

Proof. □

Problem 36. Show that

$$K := \{f \in C^1((0, 1)) \cap C^0([0, 1]) : f'(x) = |f(x)| \text{ and } |f(x)| \leq 2 \text{ holds for all } x \in (0, 1)\}$$

is a compact set when equipped with the metric

$$d(f, g) := \sup_{x \in [0, 1]} |f(x) - g(x)| + \sup_{x \in (0, 1)} |f'(x) - g'(x)|.$$

Proof. □

Problem 37. Let $h : [0, 2\pi] \rightarrow \mathbb{R}$ be continuous and $D = \{(x, y) : x^2 + y^2 \leq 1\}$. Define the sequence of functions $\{F_n\}$ on D by

$$F_n(x, y) = \int_0^{2\pi} \cos(x \sin \theta + y \cos \theta - nh(\theta)) d\theta$$

for $(x, y) \in D$ and $n \in \mathbb{N}$. Prove that $\{F_n\}$ has a subsequence that converges uniformly on D .

Proof. □

Problem 38. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be bounded and uniformly continuous. Prove that the family of functions $\{g_z\}_{z \in \mathbb{R}^n}$, $g_z(x) = f(x)f(x - z)$ is equicontinuous.

Proof. □

Problem 39. Let $\{f_n\}$ be a sequence of real valued C^1 functions on $[0, 1]$ such that, for all n ,

$$|f'_n(x)| \leq \frac{1}{\sqrt{x}} \quad (0 < x \leq 1),$$

$$\int_0^1 f_n(x) dx = 0.$$

Prove that the sequence has a subsequence that converges uniformly on $[0, 1]$.

Proof. □

Problem 40. Let the functions $f_n : [0, 1] \rightarrow [0, 1]$, $n = 1, 2, \dots$, satisfy $|f_n(x) - f_n(y)| \leq |x - y|$ whenever $|x - y| \geq 1/n$. Prove that the sequence $\{f_n\}_{n=1}^\infty$ has a uniformly convergent subsequence.

Proof. □

Problem 41. Let $\{f_n\}_{n \in \mathbb{N}} : \mathbb{R} \rightarrow \mathbb{R}$ be a sequence of continuous functions with

$$\sup_{n \in \mathbb{N}} \int_{\mathbb{R}} |f_n| < \infty.$$

Let

$$g_n(x) := \int_{\mathbb{R}} \eta(x - z) f_n(z) dz$$

for some $\eta \in C^\infty(\mathbb{R})$ and $\eta \equiv 0$ in $\mathbb{R} \setminus [-1, 1]$.

Show that

- (1) Show that $\{g_n\}_{n \in \mathbb{N}} : \mathbb{R} \rightarrow \mathbb{R}$ is uniformly bounded.
- (2) Show that $\{g_n\}_{n \in \mathbb{N}} : \mathbb{R} \rightarrow \mathbb{R}$ is equicontinuous.
- (3) Show that $\{g_n\}_{n \in \mathbb{N}}$ has a subsequence $\{g_{n_i}\}_{i \in \mathbb{N}}$ which is uniformly convergent in any finite interval (a, b)

Proof. □