

### Homework 5 for Math 1540

Due day: Sunday, February 22, Canvas.

**Problem 42.** If  $f = (f_1, \dots, f_n) : [a, b] \rightarrow \mathbb{R}^n$  is a continuous function, then we define

$$\int_a^b f(t) dt = \left\langle \int_a^b f_1(t) dt, \dots, \int_a^b f_n(t) dt \right\rangle.$$

Prove that

$$\left\| \int_a^b f(t) dt \right\| \leq \int_a^b \|f(t)\| dt.$$

*Proof.*

□

**Problem 43.** We know that every continuous function  $f : [a, b] \rightarrow \mathbb{R}$  can be uniformly approximated by polynomials (Weierstrass' theorem). Prove that if a continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  can be uniformly approximated by polynomials on all of  $\mathbb{R}$ , then  $f$  is a polynomial.

*Proof.*

□

**Problem 44.** We know that if  $f : [-1, 1] \rightarrow \mathbb{R}$  is continuous and

$$(1) \quad \int_{-1}^1 f(x) x^n dx = 0$$

for  $n = 0, 1, 2, 3, \dots$ , then  $f(x) = 0$  for all  $-1 \leq x \leq 1$ . We proved it using the Weierstrass theorem (see Example 13.48). Suppose now that  $f : [-1, 1] \rightarrow \mathbb{R}$  is continuous and (1) holds for all  $n \geq 2026$ . Prove that  $f(x) = 0$  for all  $-1 \leq x \leq 1$ .

*Proof.*

□

**Problem 45.** Prove that if  $f : [0, 1] \rightarrow \mathbb{R}$  is such that

$$\int_0^1 f(x) e^{nx} dx = 0 \quad \text{for all } n = 0, 1, 2, \dots,$$

then  $f(x) = 0$  for all  $0 \leq x \leq 1$ . Provide two proofs following the methods:

- (a) Use the Stone-Weierstrass theorem.
- (b) Use the change of variables formula and apply the Weierstrass theorem.

*Proof.*

□

**Problem 46.** Prove that the trigonometric polynomials

$$T(x) = \sum_{k=0}^n a_k \cos kx + \sum_{k=0}^n b_k \sin kx, \quad a_k, b_k \in \mathbb{R}$$

form an algebra. *Hint:*  $\cos x + i \sin x = e^{ix}$ .

*Proof.*

□

**Problem 47.** Let  $S^1 = \{z \in \mathbb{C} : |z| = 1\}$  be the unit circle in the complex plane. Let  $\mathcal{A}$  be the algebra of functions of the form

$$f(e^{i\theta}) = \sum_{n=0}^N c_n e^{in\theta}, \quad c_n \in \mathbb{C}, \theta \in \mathbb{R}.$$

It is easy to see that  $f \equiv 1$  belongs to  $\mathcal{A}$  and  $\mathcal{A}$  separates points (do not prove it). Prove that there are complex valued functions on  $S^1$  that cannot be uniformly approximated by functions in  $\mathcal{A}$ . *Hint:* For  $f \in \mathcal{A}$

$$\int_0^{2\pi} f(e^{i\theta}) e^{i\theta} d\theta = 0.$$

*Proof.*

□

**Problem 48.** Without using the theory of holomorphic functions, prove that complex polynomials

$$p(z) = \sum_{n=0}^N c_n z^n, \quad c_n \in \mathbb{C}$$

are not dense in  $C(\overline{D}, \mathbb{C})$ , where  $\overline{D} = \{z \in \mathbb{C} : |z| \leq 1\}$  is the unit disc in  $\mathbb{C}$ . *Hint:* Consider  $f(z) = \bar{z}$ . Is the previous exercise helpful?

*Proof.*

□

**Problem 49.** Prove that if a metric space  $X$  is compact, then the space  $(C(X), d_\infty)$  is separable.

**Hint:** Consider functions  $x \mapsto d(x, x_o)$  and use the Stone-Weierstrass theorem.

*Proof.*

□