

Homework 6 for Math 1540
Due day: Sunday, March 1, Canvas.

Problem 50. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \frac{x^2 y \sqrt{x^2 + y^2}}{x^4 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Prove that

- (a) f is continuous at $(0, 0)$,
- (b) all directional derivatives at $(0, 0)$ are equal 0
- (c) the function is not differentiable at $(0, 0)$.

Proof. □

Problem 51. Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) = \begin{cases} \frac{x^2(y^4 + 2x)}{x^2 + y^4} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Prove that f is differentiable at $(0, 0)$.

Proof. □

Problem 52. Prove that the function

$$f(x, y) = \begin{cases} \frac{x|y|}{\sqrt{x^2 + y^2}} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

has all directional derivatives $D_v f(0, 0)$ at the origin, but f is not differentiable at $(0, 0)$.

Proof. □

Problem 53. Prove that if $f, g : \Omega \rightarrow \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ open, are differentiable at $x_o \in \Omega$, then $fg : \Omega \rightarrow \mathbb{R}$ is differentiable at x_o and $f/g : \Omega \rightarrow \mathbb{R}$ is differentiable at x_o provided $g \neq 0$ in Ω

Proof. □

Problem 54. Let $A = [a_{ij}]$ be the matrix of a linear mapping $A \in L(\mathbb{R}^n, \mathbb{R}^m)$. Prove that the norm

$$\|A\| = \sup_{\|x\|=1} \|Ax\|$$

satisfies the inequality

$$\|A\| \leq \|A\|_2 := \left(\sum_{i=1}^m \sum_{j=1}^n a_{ij}^2 \right)^{1/2}$$

Hint: You may use the following argument: Write the components of the vector Ax as scalar products of rows on A and x . Then use the Schwarz inequality to estimate the length of the vector Ax .

Proof. □

Problem 55. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable and $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $F(x, y) = f(xy)$. Prove that

$$x \frac{\partial F}{\partial x} = y \frac{\partial F}{\partial y}.$$

Proof. □

Problem 56. We say that a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is homogeneous of degree m if $f(tx) = t^m f(x)$ for all $x \in \mathbb{R}^n$ and all $t > 0$. Prove that if f is differentiable on $\mathbb{R}^n$ and homogeneous of degree m , then

$$\sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}(x) = m f(x) \quad \text{for all } x \in \mathbb{R}^n.$$

Proof.

□

Problem 57. We know that a function $f(x, y)$ is differentiable at $(0, 0)$. We also know the directional derivatives

$$\begin{aligned} D_u f(0, 0) &= 1 & \text{where } u &= [1/\sqrt{5}, 2/\sqrt{5}], \\ D_v f(0, 0) &= 1 & \text{where } v &= [1/\sqrt{2}, 1/\sqrt{2}]. \end{aligned}$$

Find the gradient $\nabla f(0, 0)$.

Proof.

□