

Homework 7 for Math 1540

Due day: Sunday, March 15, Canvas.

Problem 58. Let $f \in C^1(\mathbb{R}^2)$ be such that $f(1,1) = 1$ and $\nabla f(1,1) = (a,b)$. Let $\varphi(x) = f(x, f(x, f(x, x)))$. Find $\varphi(1)$ and $\varphi'(1)$.

Proof. □

Problem 59. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable. Find the derivative of the function

$$F(t) = (f(t, t^2, \dots, t^n))^2, \quad t \in \mathbb{R}$$

of one variable.

Proof. □

Problem 60. Consider the function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$g(x, y) = x^{2/3}y^{2/3}, \text{ for all } (x, y) \in \mathbb{R}^2.$$

Prove that g is *differentiable* at $(0,0)$.

Proof. □

Problem 61. Find all the points $(x, y) \in \mathbb{R}^2$ where the function

$$f(x, y) = |e^x - e^y| \cdot (x + y - 2)$$

is differentiable.

Proof. □

Problem 62. Prove that if the partial derivatives (of first order) of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ exist everywhere and they are bounded, then the function f is continuous.

Proof. □

Problem 63. Prove that if $f, g \in C^k(\Omega)$, $\Omega \subset \mathbb{R}^n$, then for any multiindex α with $|\alpha| \leq k$ we have

$$D^\alpha(fg) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta f D^{\alpha-\beta} g,$$

where $\beta \leq \alpha$ means that $\beta_i \leq \alpha_i$ for $i = 1, 2, \dots, n$, $\alpha - \beta = (\alpha_1 - \beta_1, \dots, \alpha_n - \beta_n)$ and

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha - \beta)!}.$$

Proof. □

Problem 64. Prove directly from the definition (do not use the Sylvester theorem) that the matrix

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

is positive definite iff $a > 0$ and $ad - b^2 > 0$

Proof. □

Problem 65. Provide examples of functions $f \in C^2(\mathbb{R}^2)$ such that $\nabla f(0,0) = 0$, $f_{xx}(0,0) > 0$, $\mathcal{D} = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}^2(0,0) = 0$ and

- (a) f has a local minimum at $(0,0)$,
- (b) f has neither minimum nor maximum at $(0,0)$.

Prove that f cannot have a local maximum at $(0,0)$.

Proof. □

Problem 66. Prove that if $f \in C^2(\mathbb{R}^n)$, $Df(0) = 0$, and $D^2f(0)$ is positive definite, then there is $\varepsilon > 0$ such that $g(x) = f(x) - \varepsilon|x|^2$ has a local minimum at $x = 0$.

Proof. □

Problem 67. Show that if $f \in C^2(\mathbb{R}^n)$ has a local minimum at $x = 0$ and $f(0) = 0$, then

$$f(x) = \int_0^1 (1-t)x^T D^2f(tx) x dt$$

(show all details of the derivation of the above formula). Is the above formula is still true if $f(0) = 0$, but instead of a local minimum we have a local maximum?

Proof. □

Problem 68. Prove that if $f \in C^2(\mathbb{R}^n)$ and $D^2f(x)$ is positive semi-definite for all $x \in \mathbb{R}^n$, then $f(y) \geq f(x) + Df(x)(y - x)$ for all $x, y \in \mathbb{R}^n$.

Proof. □

Problem 69. Let $n \geq 2$.

(a) Show that there is a non-constant map $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ of class C^∞ such that

(1)
$$\langle F(x) - F(y), x - y \rangle = 0 \quad \text{for all } x, y \in \mathbb{R}^n.$$

(b) Show that if $f \in C^\infty(\mathbb{R}^n)$ and $F = \nabla f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies (1), then F is constant.

Hint. You can use the following fact from linear algebra: If B is a symmetric matrix, then the operator norm satisfies $\|B\| = \sup_{|v|=1} \langle Bv, v \rangle$.

Proof. □