Analysis 1: homework # 1

Due day: Friday September 18, 2015

NAME (print):

Circle the problems that you have solved:

1 2 3 4 5 6 7

The solutions must be written in a **legible** form. The front page **must** be returned. All the papers **must** be stapled.

Problem 1. Prove that $\mathfrak{M}$ is a σ-algebra of subsets of X if and only is

(1) $X \in \mathfrak{M}$,
(2) If $A, B \in \mathfrak{M}$, then $A \setminus B \in \mathfrak{M}$,
(3) If $A_1, A_2, A_3 \ldots \in \mathfrak{M}$ are pairwise disjoint, then $\bigcup_{i=1}^{\infty} A_i \in \mathfrak{M}$.

Problem 2. Let X be a metric space and let μ be a measure in $\mathfrak{B}(X)$. Prove that

$$\mu^*(E) = \inf_{\substack{U \supset E \\ U-\text{open}}} \mu(U)$$

defines a metric outer measure.

Problem 3. Prove that $\mathcal{L}_n^*$ is a metric outer measure.

Problem 4. Show that

- If $\mathcal{H}^s(E) < \infty$, then $\mathcal{H}^t(E) = 0$ for all $t > s$;
- If $\mathcal{H}^s(E) > 0$, then $\mathcal{H}^t(E) = \infty$ for all $0 < t < s$.

Problem 5. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a homeomorphism. Prove that $A \subset \mathbb{R}^n$ is Borel if and only if $f(A)$ is Borel.

Problem 6. Let $\mathfrak{C} \subset [0, 1]$ be the ternary Cantor set. Prove that

(1) $\mathcal{L}_1(\mathfrak{C}) = 0$,
(2) $\dim_H(\mathfrak{C}) \leq \log 2/\log 3$.

Problem 7. Prove that the graph G_f of a continuous function $f : \mathbb{R} \to \mathbb{R}$ as a subset of $\mathbb{R}^2$ has two dimensional Lebesgue measure zero $\mathcal{L}_2(G_f) = 0$.