Analysis 1: homework # 3

Due day: Friday October 16, 2015

NAME (print):

Circle the problems that you have solved:

15 16 17 18 19 20 21 22 23 24

The solutions must be written in a **legible** form. The front page **must** be returned. All the papers **must** be stapled.

Problem 15. Let ℓ^∞ be a metric space of all real bounded sequences $x = (x_1, x_2, x_3, \ldots)$ with the metric $\|x - y\|_\infty = \sup_i |x_i - y_i|$. Prove that there is an uncountable family of pairwise disjoint balls in ℓ^∞ (and hence ℓ^∞ is not separable). **Hint:** *You can use (without a proof) the fact that the family of subsets of* $\mathbb{N}$ *is uncountable.*

Problem 16. Let $K \subset \mathbb{R}^n$ be compact and let $K_\varepsilon = \{x : \mathrm{dist}(x, K) < \varepsilon)$. Prove that $|K_\varepsilon| \to |K|$ as $\varepsilon \to 0$. Show an example of a non-compact set for which this statement is no longer true.

Problem 17. Let $U \subset \mathbb{R}^n$ be an open set. Prove that there is a sequence of pairwise disjoint closed balls $\overline{B}_i \subset U$, $i = 1, 2, 3, \ldots$ such that $|U \setminus \bigcup_{i=1}^\infty \overline{B}_i| = 0$.

Problem 18. For $f : [0, 1] \to \mathbb{R}$ let $E \subset \{x : f'(x) \text{ exists}\}$. Prove that if $|E| = 0$, then $|f(E)| = 0$.

Problem 19. Show that for any Lebesgue measurable set $E \subset \mathbb{R}$ with $|E| = 1$ there is a Lebesgue measurable subset $A \subset E$ such that $|A| = \frac{1}{2}$.

Problem 20. Let $A \subset [0, 1]$ be a measurable set of positive measure. Show that there exist two points $x, y \in A$, $x \neq y$ such that $x - y$ is a rational number.

Problem 21. Suppose that $f_n : X \to [0, \infty]$ is a sequence of measurable functions such that $f_1 \in L^1(\mu)$ and $f_1 \geq f_2 \geq \ldots \geq 0$, $f_n(x) \to f(x)$ for every $x \in X$. Prove that

$$\lim_{n\to\infty} \int_X f_n(x)\, d\mu = \int_X f(x)\, d\mu.$$

Show also that the assumption $f_1 \in L^1(\mu)$ cannot be removed.

Problem 22. Prove that if f_n converges in measure to f and converges in measure to g, then $f = g$ a.e.

Problem 23. Show an example of a sequence of measurable functions $f_n : [0, 1] \to \mathbb{R}$ that converge to the function $f = 0$ in measure but not a.e.

Problem 24. Show an example of a sequence $f_n : \mathbb{R} \to \mathbb{R}$ of measurable functions that converge of a function f a.e. but not in measure.