Analysis 1: homework # 1
Due day: Friday August 28, 2020

NAME (print):

If you do **not** have a complete solution do not submit it as you will get negative points for an incomplete solution. All solutions have to be written in LaTeX using this template and submitted as a pdf file.

Problem 1. Prove that $\mathfrak{M}$ is a σ-algebra of subsets of X if and only is

(1) $X \in \mathfrak{M}$,
(2) If $A, B \in \mathfrak{M}$, then $A \setminus B \in \mathfrak{M}$,
(3) If $A_1, A_2, A_3 \ldots \in \mathfrak{M}$ are pairwise disjoint, then $\bigcup_{i=1}^{\infty} A_i \in \mathfrak{M}$.

Proof. (Write your solution here.) □

Problem 2. Use counting measure μ to show that it can happen that $A_1 \supset A_2 \supset A_3 \supset \ldots$, but

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) \neq \lim_{i\to\infty} \mu(A_i).$$

Proof. (Write your solution here.) □

Problem 3. Let X be a metric space and let μ be a measure in $\mathfrak{B}(X)$. Prove that

$$\mu^*(E) = \inf_{\substack{U \supset E \\ U-\text{open}}} \mu(U)$$

defines a metric outer measure.

Proof. (Write your solution here.) □

Problem 4. Show that

- If $\mathcal{H}^s(E) < \infty$, then $\mathcal{H}^t(E) = 0$ for all $t > s$;
- If $\mathcal{H}^s(E) > 0$, then $\mathcal{H}^t(E) = \infty$ for all $0 < t < s$.

Proof. (Write your solution here.) □

Problem 5. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a homeomorphism. Prove that $A \subset \mathbb{R}^n$ is Borel if and only if $f(A)$ is Borel.

Proof. (Write your solution here.) □

Problem 6. Let $\mathfrak{C} \subset [0,1]$ be the ternary Cantor set. Prove that $\dim_H(\mathfrak{C}) \leq \log 2/\log 3$.

Proof. (Write your solution here. I did explain the proof in a lecture, but it is only in the video so you need to understand it and the best way to do it is to write your solution. It is okay if you copy my argument.) □

Problem 7. Prove that the set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ is a Borel set.

Proof. (Write your solution here.) $\square$