Analysis 1: homework # 3
Due day: Friday September 18, 2020

NAME:

If you do **not** have a complete solution do not submit it as you will get negative points for an incomplete solution. All solutions have to be written in LaTeX using this template and submitted as a pdf file.

Problem 14. Prove that if X is a locally compact separable metric space, then there are open sets $\{U_i\}_{i=1}^{\infty}$ such that

$$X = \bigcup_{i=1}^{\infty} U_i \quad \text{and} \quad \overline{U}_i \text{ is compact.}$$

Proof. (Write your solution here.) □

Problem 15. Does there exist an enumeration $\{r_n\}_{n=1}^{\infty}$ of the rationals, such that the complement of

$$\bigcup_{n=1}^{\infty} \left(r_n - \frac{1}{n}, r_n + \frac{1}{n} \right)$$

is $\mathbb{R}$ is non-empty?

Proof. (Write your solution here.) □

Problem 16. Let A be the subset of $[0, 1]$ which consists of all numbers which do not have the digit 4 appearing in their decimal expansion. Find $\mathcal{L}_1(A)$.

Proof. (Write your solution here.) □

Problem 17. Suppose $\{E_k\}_{k=1}^{\infty}$ is a countable family of measurable subsets of $\mathbb{R}^n$ such that $\sum_{k=1}^{\infty} \mathcal{L}_n(E_k) < \infty$ and let

$$E = \{x \in \mathbb{R}^n : x \in E_k \text{ for infinitely many } k\}.$$

Show that E is measurable and that $\mathcal{L}_n(E) = 0$.

Proof. (Write your solution here.) □

Problem 18. Let $K \subset \mathbb{R}^2$ be a compact set and let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the orthogonal projection onto the x-axis. Prove that if $\pi(K) = [0, 1]$, then $\mathcal{H}^1(K) \geq 1$.

Proof. (Write your solution here.) □

Problem 19. You can use the fact that if $S^1 \subset \mathbb{R}^2$ is the unit circle, then $\mathcal{H}^1(S^1) = 2\pi$. Assume that $\phi : S^1 \to \mathbb{R}^2$ is a map such that for some $L \geq 1$ and $0 < s < 1$ we have

$$\frac{1}{L}|x - y|^s \leq |\phi(x) - \phi(y)| \leq L|x - y|^s \quad \text{for all } x, y \in S^1.$$

Find $\dim_H(\phi(S^1))$.

Proof. (Write your solution here.) □

Problem 20. Regard $\mathbb{R}$ as a subset of $\mathbb{R}^2$ (the x-axis). Prove that $A \subset \mathbb{R}$ is Borel as a subset of $\mathbb{R}$ if and only if it is Borel as a subset of $\mathbb{R}^2$.

Proof. (Write your solution here.) □

Problem 21. Prove that there is an $\mathcal{L}_2$ measurable set $A \subset \mathbb{R}^2$ that is not Borel.

Proof. (Write your solution here.) □

Problem 22. Prove that if $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ are $\mathcal{L}_n$ and $\mathcal{L}_m$ measurable respectively, then $A \times B$ is $\mathcal{L}_{n+m}$ measurable.

Proof. (Write your solution here.) □

Problem 23. Prove that if $A \subset \mathbb{R}^n$ satisfies $\mathcal{L}_n(A) = 0$ and $B \subset \mathbb{R}^m$ is any set, then $\mathcal{L}_{n+m}(A \times B) = 0$.

Proof. (Write your solution here.) □

Problem 24. Prove that the graph

$$G_f = \{(x, f(x)) : y = f(x)\} \subset \mathbb{R}^2$$

of a continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfies $\mathcal{L}_2(G_f) = 0$.

Proof. (Write your solution here.) □