Analysis 1: homework # 4
Due day: Friday September 25, 2020

NAME:

If you do **not** have a complete solution do not submit it as you will get negative points for an incomplete solution. All solutions have to be written in LaTeX using this template and submitted as a pdf file.

Problem 25. Prove that if X is a compact metric space, then for any $\varepsilon > 0$ and any $s > 0$, $\mathcal{H}^s_\varepsilon(X) < \infty$.

Proof. (Write your solution here.) □

Problem 26. Prove that if X is a bounded metric space and $\varepsilon > \operatorname{diam} X$, then for any $s > 0$, $\mathcal{H}^s_\varepsilon(X) < \infty$.

Proof. (Write your solution here.) □

Problem 27. Prove that there is a bounded separable metric space such that for any $0 < \varepsilon < \operatorname{diam} X$ and any $s > 0$, $\mathcal{H}^s_\varepsilon(X) = \infty$.
Hint: *Subset of ℓ^∞. Be careful. Must be separable.*

Proof. (Write your solution here.) □

Problem 28. Prove that is a set $E \subset \mathbb{R}$ has positive measure $|E| > 0$, then the set

$$E - E = \{x - y : \, x, y \in E\}$$

contains an interval.
Hint: *The Steinhaus theorem.*

Proof. (Write your solution here.) □

Problem 29. Prove that if $K \subset \mathbb{R}^n$ is compact and $K_\varepsilon = \{x : \, \operatorname{dist}(x, K) < \varepsilon\}$, then $|K_\varepsilon| \to |K|$ as $\varepsilon \to 0$. Show an example of a non-compact measurable set such that $|K_\varepsilon| \not\to |K|$.

Proof. (Write your solution here.) □

Problem 30. Show that there is a function $f : \mathbb{R} \to \mathbb{R}$ that is not Lebesgue measurable, but it has the property that for every $a \in \mathbb{R}$, the set $f^{-1}(a)$ is Lebesgue measurable. **Hint:** *Use in the construction of f the Vitali non-measurable set.*

Proof. (Write your solution here.) □

Problem 31. Give an example of a measurable function $f : [0, 1] \to [0, \infty)$ (i.e., with finite values) such that for all $0 < a < b < 1$ we have

$$\int_a^b f(x)\, dx = \infty.$$

Proof. (Write your solution here.) ☐

Problem 32. Prove that if $f : X \to [0, \infty]$ is measurable and $\int_X f \, d\mu < \infty$, then $\mu(\{x : f(x) = +\infty\}) = 0$.

Proof. (Write your solution here.) ☐

Problem 33. Prove that if the functions $f, g : X \to \mathbb{R}$ are measurable, then the set $\{x : f(x) = g(x)\}$ is measurable.

Proof. (Write your solution here.) ☐

Problem 34. Construct a sequence of measurable functions $f_n : \mathbb{R} \to [0, \infty]$ such that we do not have equality in Fatou's lemma.

Proof. (Write your solution here.) ☐

Problem 35. Prove the following stronger version of the Lebesgue dominated convergence theorem:
Suppose that $\{f_n\}$ is a sequence of complex measurable functions on X such that

$$f(x) = \lim_{n\to\infty} f_n(x)$$

for every $x \in X$. Suppose that there is a sequence of functions $g_n \in L^1(\mu)$ and a function $g \in L^1(\mu)$ such that

$$|f_n(x)| \le g_n(x) \qquad (n = 1, 2, 3, \dots; x \in X)$$

and

$$\lim_{n\to\infty} \int_X |g_n - g| \, d\mu = 0.$$

Then $f \in L^1(\mu)$ and

$$\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0.$$

Proof. (Write your solution here.) ☐