Analysis 1: homework # 5

Due day: Friday October 2+ε, 2020, for some $\varepsilon \geq 0$ (possibly large)

NAME:

If you do **not** have a complete solution do not submit it as you will get negative points for an incomplete solution. All solutions have to be written in LaTeX using this template and submitted as a pdf file.

Problem 36. Let $U \subset \mathbb{R}^n$ be an open set. Prove that there is a sequence of pairwise disjoint closed balls $\overline{B}_i \subset U$, $i = 1, 2, 3, \dots$ such that $|U \setminus \bigcup_{i=1}^{\infty} \overline{B}_i| = 0$.

Proof. (Write your solution here.) □

Problem 37. For $f : [0,1] \to \mathbb{R}$ let $E \subset \{x : f'(x) \text{ exists}\}$. Prove that if $|E| = 0$, then $|f(E)| = 0$.

Proof. (Write your solution here.) □

Problem 38. Show that for any Lebesgue measurable set $E \subset \mathbb{R}$ with $|E| = 1$ there is a Lebesgue measurable subset $A \subset E$ such that $|A| = \frac{1}{2}$.

Proof. (Write your solution here.) □

Problem 39. Let $A \subset [0,1]$ be a measurable set of positive measure. Show that there exist two points $x, y \in A$, $x \neq y$ such that $x - y$ is a rational number.
(Prove it directly without using Steinhaus' theorem).

Proof. (Write your solution here.) □

Problem 40. Suppose that $f_n : X \to [0, \infty]$ is a sequence of measurable functions such that $f_1 \in L^1(\mu)$ and $f_1 \geq f_2 \geq \ldots \geq 0$, $f_n(x) \to f(x)$ for every $x \in X$. Prove that

$$\lim_{n\to\infty} \int_X f_n(x)\, d\mu = \int_X f(x)\, d\mu.$$

Show also that the assumption $f_1 \in L^1(\mu)$ cannot be removed.

Proof. (Write your solution here.) □

Problem 41. Let $\mathfrak{M}$ be a σ-algebra of subsets of X that contains countably many nonempty pairwise disjoint sets. Prove that $\mathfrak{M}$ is uncountable.

Proof. (Write your solution here.) □

Problem 42. Prove that if $\mathfrak{M}$ is an infinite σ-algebra, then it is uncountable.
Remark. *Note that the assumptions are weaker than in Problem 41 and clearly Problem 41 follows from Problem 42.*

Proof. (Write your solution here.) □

Problems 43-46 provide a proof that the Vitali set is not Borel without using the Lebesgue measure.

Definition. A set in a metric space is *nowhere dense* if its closure has no interior points. A set that is a union of countably many nowhere dense sets is called a *set of first category.* A set that is not of first category is called a set of *second category.*

Definition. A subset A of a metric space is called *almost open* if there is an open set U such that the symmetric difference $A\Delta U$ is of first category i.e., it is a union of countably many nowhere dense sets.

Problem 43. Prove that if X is a complete metric space, then X is of the second category.
Hint. *This is just the Baire category theorem.*

Proof. (Write your solution here.) □

Problem 44. Prove that a family of almost open sets in a metric spaces is a σ-algebra.

Proof. (Write your solution here.) □

Problem 45. Prove that every Borel set is almost open.

Proof. (Write your solution here.) □

Problem 46. Prove that the Vitali set is not almost open and hence it is not Borel.

Proof. (Write your solution here.) □

Problem 47. Prove that if $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ are Lebesgue measurable, then

$$\mathcal{L}_{n+m}(A \times B) = \mathcal{L}_n(A)\mathcal{L}_m(B).$$

Proof. (Write your solution here.) □

Problem 48. Prove that there is a *finitely additive* measure defined on all subsets of positive integers $\mathbb{N}$, with values into $\{0,1\}$, (only two values) that is

$$\mu : 2^{\mathbb{N}} \to \{0,1\}$$

such that $\mu(\{n\}) = 0$ for every $n \in \mathbb{N}$ and $\mu(\mathbb{N}) = 1$.
Hint. *Non-principal ultrafilter. Clearly μ cannot be countably additive as $\mu(\{n\}) = 0$ would imply that $\mu(\mathbb{N}) = 0$.*

Proof. (Write your solution here.) □

Definition. We say that a function $\phi : \mathbb{R} \to \mathbb{R}$ is *Cauchy* if

$$\phi(x+y) = \phi(x) + \phi(y) \quad \text{for all } x, y \in \mathbb{R}. \tag{1}$$

Clearly linear functions $\phi(x) = ax$ are Cauchy.

Problem 49. Prove that if a Cauchy function is continuous, then it is linear.

Proof. (Write your solution here.) ☐

Problem 50. Prove that if a Cauchy function is continuous at 0, then it is linear.

Proof. (Write your solution here.) ☐

Problem 51. Prove that there is a Cauchy function such that $\phi(1) = 0$ and $\phi(\pi) = 1$.
Hint. *Clearly, this functions cannot be linear. To prove existence of ϕ regard $\mathbb{R}$ as a linear space over the field of rational numbers $\mathbb{Q}$ and use the Hamel basis.*

Proof. (Write your solution here.) ☐

Problem 52. Prove that is a Cauchy function is not linear, then it is not Lebesgue measurable.
Hint. *Use the Steinhaus theorem or use the Lusin theorem.*

Proof. (Write your solution here.) ☐