Analysis 1: homework # 6
Due day: Friday October 23, 2020.

NAME:

If you do **not** have a complete solution do not submit it as you will get negative points for an incomplete solution. All solutions have to be written in LaTeX using this template and submitted as a pdf file.

Problem 53. Prove that if f_n converges in measure to f and converges in measure to g, then $f = g$ a.e.

Proof. (Write your solution here.) □

Problem 54. Show an example of a sequence of measurable functions $f_n : [0, 1] \to \mathbb{R}$ that converge to the function $f = 0$ in measure but not a.e.

Proof. (Write your solution here.) □

Problem 55. Show an example of a sequence $f_n : \mathbb{R} \to \mathbb{R}$ of measurable functions that converge of a function f a.e. but not in measure.

Proof. (Write your solution here.) □

Problem 56. Suppose that $f_n \to f$ in measure and $g_n \to g$ in measure. Show that $f_n g_n \to fg$ in measure if $\mu(X) < \infty$. Show that the claim is not necessarily true if $\mu(X) = \infty$.

Proof. (Write your solution here.) □

Problem 57. Let μ be the counting measure on the set of positive integers $\mathbb{N}$ and let $f_n, f : \mathbb{N} \to \mathbb{R}$ be given functions. Prove that $f_n \to f$ in measure as $n \to \infty$ if and only if $f_n \to f$ uniformly.

Proof. (Write your solution here.) □

Problem 58. Let X be a metric space and let μ be a σ-finite measure on $\mathfrak{B}(X)$. Prove that for every $x \in X$ and almost all $r > 0$

$$\mu(S(x, r)) = 0, \quad \text{where} \quad S(x, r) = \{z \in X : d(z, x) = r\}.$$

Proof. (Write your solution here.) □

Problem 59. Show an example of a continuous function between metric spaces $f : X \to Y$ such that the function $N_f : Y \to [0, \infty]$ is not Borel, where $N_f(y)$ is the number of points in the set $f^{-1}(y)$.

Proof. (Write your solution here.) □

Problem 60. Let $f : X \to Y$ be a continuous an surjective map between compact metric spaces. Prove that there is a Borel set $B \subset X$ such that $f(B) = Y$, f is one-to-one on B and $f^{-1} : Y \to B$ is Borel.
Remark. *In other words, we can select from each of the sets f^{-1} exactly one point in a way that the resulting inverse function is Borel.*

Proof. (Write your solution here.) □

Problem 61. Prove that if Y is a compact metric space, then the metric space of continuous functions $C(Y)$ is separable.
Hint. *Use the Stone-Weierstrass theorem and distance functions $y \mapsto d(y_o, y)$.*

Proof. (Write your solution here.) □

Problem 62. Let X be a compact metric space and μ a finite measure on $\mathfrak{B}(X)$. Let Y be another compact metric space. Assume that a function $f : X \times Y \to \mathbb{R}$ has the following properties

(a) For every $y \in Y$,
$$f^y(\cdot) = f(\cdot, y) : X \to \mathbb{R} \quad \text{is Borel;}$$
(b) For every $x \in X$,
$$f_x(\cdot, y) : Y \to \mathbb{R}$$
is continuous.

Prove that for any $\varepsilon > 0$ there is a compact set $K \subset X$ such that $\mu(X \setminus K) < \varepsilon$ and the restriction of f to $K \times Y$,
$$f : K \times Y \to \mathbb{R}$$
is continuous.
Hint. *Use previous problem.*
Remark. *Similar functions often appear in the calculus of variations. The above result can be regarded as a generalization of the Lusin theorem.*

Proof. (Write your solution here.) □

Problem 63. Prove that if X and Y are metric spaces, then $\mathfrak{B}(X) \times \mathfrak{B}(Y) \subset \mathfrak{B}(X \times Y)$. Prove that if in addition, the metric spaces X and Y are separable, then $\mathfrak{B}(X) \times \mathfrak{B}(Y) = \mathfrak{B}(X \times Y)$.

Proof. (Write your solution here.) □

Problem 64. Show an example that the Fubini theorem is not true if the measures are not σ-finite.

Proof. (Write your solution here.) □

Problem 65. Prove that if $f : \mathbb{R}^n \to \mathbb{R}$ is Lebesgue measurable, then the graph of f,
$$G_f = \{(x, f(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$$
has $(n + 1)$-dimensional Lebesgue measure zero.

Proof. (Write your solution here.) □

Problem 66. Prove that if $f : \mathbb{R}^n \to [0, \infty)$ is Lebesgue measurable, then the set

$$U_f = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : x \in \mathbb{R}^n,\ 0 \le y \le f(x)\} \subset \mathbb{R}^{n+1}$$

is Lebesgue measurable and

$$\mathcal{L}_{n+1}(U_f) = \int_{\mathbb{R}^n} f \, d\mathcal{L}_n.$$

Proof. (Write your solution here.) □

Definition. Let (X, μ) be a complete measure space. For a function $f : X \to [0, \infty]$ defined μ-a.e. on X, the *upper integral* is defined by

$$\int_X^* f \ d\mu = \inf \int_X \phi \, d\mu \,,$$

where the infimum is taken over all *μ-measurable* functions ϕ satisfying $0 \le f(x) \le \phi(x)$ for μ-a.e. $x \in X$. We do note require f to be measurable. Clearly, for measurable functions the upper integral coincides with the Lebesgue one.

Problem 67. Prove that if $\int_X^* f \, d\mu = 0$, then $f = 0$ μ-a.e. and hence it is measurable.

Proof. (Write your solution here.) □

Problem 68. (Non-measurable monotone convergence theorem.) Let $f_n : X \to [0, \infty]$ be a monotone sequence of (not necessarily measurable) functions, i.e. $0 \le f_1(x) \le f_2(x) \le \ldots$ for μ-a.e. $x \in X$. Prove that if $f(x) := \lim_{n\to\infty} f_n(x)$, then

$$\lim_{n\to\infty} \int_X^* f_n \, d\mu = \int_X^* f \, d\mu \,. \tag{1}$$

Proof. (Write your solution here.) □

Definition. We say $\phi : X \to [0, \infty]$ is a *step function* if it is μ-measurable and attains at most countably many values (we allow infinite values). That is, ϕ is a step function if there exist disjoint μ-measurable subsets $A_i \subset X$ and $0 \le a_i \le \infty$ such that

$$\phi(x) = \sum_{i=1}^{\infty} a_i \chi_{A_i}(x) \,.$$

Note that the difference with the definition of a simple function is that a simple function is finite and has only finitely many values.

Problem 69. Let $f : X \to [0, \infty]$ be any function. Prove that

$$\int_X^* f \ d\mu = \inf \int_X \phi \, d\mu \,,$$

where the infimum is over all step functions ϕ satisfying $0 \le f(x) \le \phi(x)$ for all $x \in X$.

Proof. (Write your solution here.) □

Problem 70. Let $\mathcal{F}$ be any family of Lebesgue measurable functions $f : \mathbb{R}^n \to \mathbb{R}$. Prove that there is a Lebesgue measurable function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with the following two properties.

(a) For all $f \in \mathcal{F}$ we have $f \leq g$ a.e.
(b) If $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is Lebesgue measurable and for all $f \in \mathcal{F}$ we have $f \leq h$ a.e., then $g \leq h$ a.e.

Remark. In the case in which family $\mathcal{F}$ is countable, we can define g as the supremum of all functions in the family $\mathcal{F}$. However, if the family is uncountable, taking supremum does not make much sense.
Hint. To make the problem easier you can assume that the functions are defined on a subset of $\mathbb{R}^n$ of finite measure and that all functions are uniformly bounded: $|f| \leq M$ for all $f \in \mathcal{F}$. Then define g as a carefully selected countable subfamily of $\mathcal{F}$.

Proof. (Write your solution here.) $\square$

Problem 71. Prove without using any linear algebra that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry, then for any set $A \subset \mathbb{R}^n$,

$$\mathcal{L}_n^*(T(A)) = \mathcal{L}_n^*(A).$$

Hint. *Use Problem 36.*

Proof. (Write your solution here.) $\square$