Analysis 1: homework # 7
Due day: Wednesday November 18, 2020.

NAME:

If you do **not** have a complete solution do not submit it as you will get negative points for an incomplete solution. All solutions have to be written in LaTeX using this template and submitted as a pdf file.

Problem 72. Prove directly from the definition of the Lebesgue measure and the definition of the integral that if $u \in L^1(\mathbb{R}^n)$ (with respect to the Lebesgue measure), then for every $y \in \mathbb{R}^n$ and $t > 0$

$$\int_{\mathbb{R}^n} u(x)\,dx = \int_{\mathbb{R}^n} u(-x)\,dx = \int_{\mathbb{R}^n} u(x+y)\,dx = t^{-n}\int_{\mathbb{R}^n} u(x/t)\,dx.$$

Proof. (Write your solution here.) □

Problem 73. Prove directly from the definition of the Lebesgue measure and the definition of the integral that

$$\int_{\mathbb{R}^n\setminus B(0,1)} \frac{dx}{|x|^\alpha} < \infty$$

if and only if $\alpha > n$.

Proof. (Write your solution here.) □

Problem 74. Prove directly from the definition of the Lebesgue measure and the definition of the integral (you can use Problem 72) that

$$\int_{|x|\geq r} \frac{dx}{|x|^{n+1}} = \frac{C(n)}{r},$$

where $C(n)$ is a constant that depends on n only.

Proof. (Write your solution here.) □

Problem 75. Let $a_i > 0$, $1 \leq i \leq n$. Prove that

$$\int_{[0,1]^n} \frac{dx}{x_1^{a_1} + \ldots + x_n^{a_n}} < \infty \quad \text{if and only if} \quad \sum_{i=1}^n \frac{1}{a_i} > 1.$$

Proof. (Write your solution here.) □

Problem 76. Let μ be the counting measure on $\mathbb{R}$. Then $\mu \times \mathcal{L}_1$ is a measure on $\mathbb{R}^2$. Let S be the unit circle in $\mathbb{R}^2$, and χ_S its characteristic function. Prove that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \chi_S\,d\mathcal{L}_1\,d\mu \neq \int_{\mathbb{R}}\int_{\mathbb{R}} \chi_S\,d\mu\,d\mathcal{L}_1\,.$$

Explain why it is not a contradiction with the Fubini theorem.

Proof. (Write your solution here.) □

Problem 77. Prove that
$$\int_{\mathbb{R}^n} |f(x)|^{2020}\,dx = 2020\int_0^\infty t^{2019}|\{x : |f(x)| > t\}|\,dt.$$

Proof. (Write your solution here.) □

Problem 78. Integrate the function $xe^{-x^2(1+y^2)}$ over $(0,\infty)\times(0,\infty)$ in two different ways and conclude that
$$\int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}.$$

Proof. (Write your solution here.) □

Problem 79. By integrating e^{-xy} over an appropriate region, show that
$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx = \log\frac{b}{a}, \quad a, b > 0.$$

Proof. (Write your solution here.) □

Problem 80. Prove that if
$$f(x,t) = \frac{x^2-y^2}{(x^2+y^2)^2},$$
then $f \notin L^1([0,1]\times[0,1])$.

Proof. (Write your solution here.) □

Problem 81. Provide an example of a sequence $f_n \in L^p([0,1])$, $1 \le p < \infty$ which converges a.e. but not in L^p to a function $f \in L^p([0,1])$.

Proof. (Write your solution here.) □

Problem 82. Provide an example of a sequence $f_n \in L^p([0,1])$, $1 \le p < \infty$ which converges in L^p but not a.e. to a function $f \in L^p([0,1])$.

Proof. (Write your solution here.) □

The integral average is often denoted as a barred integral:
$$⨍_X f\,d\mu = \mu(X)^{-1}\int_X f\,d\mu.$$

Problem 83. Prove that if $0 < \mu(X) < \infty$, then $\left(⨍_X |f|^p\,d\mu\right)^{1/p} \le \left(⨍_X |f|^q\,d\mu\right)^{1/q}$ for all $0 < p < q < \infty$.

Proof. (Write your solution here.) □

Problem 84. Prove that if $f \in L^r$ for some $r < \infty$, then $\lim_{p\to\infty}\|f\|_p = \|f\|_\infty$.

Proof. (Write your solution here.) □

Problem 85. Prove that if $\mu(X) = 1$ and $\|f\|_r < \infty$ for some $r > 0$, then

$$\lim_{p\to 0} \|f\|_p = \exp\left(\int_X \log|f|\, d\mu\right)$$

where $\exp(-\infty)$ is defined to be 0.

Proof. (Write your solution here.) □

Problem 86. Let $f \geq 0$, $g \geq 0$ belong to L^p for some $0 < p < 1$. Prove that $\|f + g\|_p \geq \|f\|_p + \|g\|_p$.

Proof. (Write your solution here.) □

Problem 87. Prove that if

$$\int_{\mathbb{R}^n} \big|\log|f_k|\big|\, dx \to 0, \quad \text{as } k \to \infty,$$

then there is a subsequence f_{k_i} such that $|f_{k_i}| \to 1$ a.e.

Proof. (Write your solution here.) □

Problem 88. Suppose that $p, q, r \in [1, \infty)$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Prove that

$$\|fg\|_r \leq \|f\|_p \|g\|_q.$$

Proof. (Write your solution here.) □

Problem 89. Prove that if $f \in L^p(\mu)$, $1 < p < \infty$, then

$$\|f\|_p = \sup\left\{\int_X fg\, d\mu : \|g\|_q = 1\right\},$$

where q is the Hölder conjugate exponent.

Proof. (Write your solution here.) □

Problem 90. Suppose that $1 \leq p < r < q < \infty$. Prove that $L^p \cap L^q \subset L^r$.

Proof. (Write your solution here.) □

Problem 91. Prove that if $f, f_k \in L^1(\mu)$, $f_k \to f$ a.e. and $\|f\|_1 \to \|f\|_1$, then $\|f_k - f\|_1 \to 0$.

Proof. (Write your solution here.) □

Problem 92. Let $1 \leq p < \infty$. Suppose that $f_k \in L^p(\mu)$ converges μ-a.e. to $f \in L^p(\mu)$. Prove that

$$\lim_{k\to\infty} \|f_k - f\|_p = 0 \quad \text{if and only if} \quad \|f\|_p = \lim_{k\to\infty} \|f\|_p.$$

Proof. (Write your solution here.) □

Problem 93. Suppose that $1 < p < \infty$, $f, g \in L^p(\mu)$ (complex valued), $\|f\|_p, \|g\|_p > 0$ and

$$\|f + g\|_p = \|f\|_p + \|g\|_p.$$

Prove that

$$\frac{f}{\|f\|_p} = \frac{g}{\|g\|_p} \quad \text{a.e.}$$

Hint. *Look for solutions online. Read and understand them and write in your own words with all details.*

Proof. (Write your solution here.) □

Problem 94. Let $\tau_y f(x) = f(x + y)$. Prove that if $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}^n)$, then

$$\lim_{|y|\to\infty} \|\tau_y f + f\|_p = 2^{1/p}\|f\|_p.$$

Hint. *Compare with Lemma 93 in my notes.*

Proof. (Write your solution here.) □

Problem 95. Let a function f be μ-measurable with the property that there is $M > 0$ such that $|f| \leq M$ a.e. Prove that if

$$\|f\|_\infty = \inf\{M > 0 : |f| \leq M \text{ a.e.}\},$$

then $|f| \leq \|f\|_\infty$ a.e.

Proof. (Write your solution here.) □

Problem 96. Prove that $\big| \|x\| - \|y\| \big| \leq \|x - y\|$ in any normed space.

Proof. (Write your solution here.) □