Analysis 1: homework # 8

Final exam. You need to solve 10 problems.

NAME:

All solutions have to be written in LaTeX using this template and submitted as a pdf file.

Problem 97. Let f be the function on $[0,1]$ defined as follows: $f(x) = 0$ if x is a point in the Cantor set and $f(x) = 5^{-n}$ if x is in one of the intervals of length 3^{-n} that were removed in the n-th step of the construction of the Cantor set. Prove that the function $f(x)$ is measurable and evaluate the integral $\int_0^1 f(x)\, dx$.

Proof. (Write your solution here.) □

Problem 98. For $1 \leq p < \infty$, let $X^p(\mathbb{R}^n)$ be the space of functions $f \in L^p(\mathbb{R}^n)$ such that there is $0 \leq g \in L^p(\mathbb{R}^n)$ such that

$$|f(x) - f(y)| \leq |x-y|(g(x) + g(y)) \quad \text{a.e.,} \tag{1}$$

meaning that there is $N \subset \mathbb{R}^n$, $|N| = 0$ so that (1) is true for all $x, y \in \mathbb{R}^n \setminus N$. Let

$$\|f\|_{X^p} = \|f\|_p + \inf \|g\|_p,$$

where the infimum is taken over all $0 \leq g \in L^p(\mathbb{R}^n)$ satisfying (1). Prove that

(a) X^p is a linear space,
(b) $\|\cdot\|_{X^p}$ is a norm,
(c) X^p is a Banach space.

Hint: Mimic the proof that L^p is a Banach space.

Proof. (Write your solution here.) □

Problem 99. For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\lambda > 0$ we define

$$\mathfrak{M}_\lambda f(x) = \sup_{r>0} r^\lambda ⨍_{B(x,r)} |f(z)|\, dz.$$

Prove that if $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, and $\lambda p < n$, then

$$\mathcal{H}^{n-\lambda p}_\infty(\{x : \mathfrak{M}_\lambda f(x) > t\}) \leq \frac{C(n,p,\lambda)}{t^p} \int_{\mathbb{R}^n} |f(z)|^p\, dz \quad \text{for all } t > 0.$$

Hint: Use Hölder's inequality and mimic the proof of the Hardy-Littlewood-Wiener theorem.

Proof. (Write your solution here.) □

Problem 100. Prove that no Lebesgue measurable set $E \subset \mathbb{R}$ has the property that $|E \cap [a,b]| = (b-a)/2$ for all $a < b$.

Proof. (Write your solution here.) □

Problem 101. Let $K \subset \mathbb{R}^n$ be a compact set and for $r > 0$ let

$$K_r = \{x \in \mathbb{R}^n : \operatorname{dist}(x, K) = r\}.$$

Prove that for any $r > 0$, the set K_r has measure zero, $|K_r| = 0$. *Hint:* Look at the density points.

Proof. (Write your solution here.) □

Problem 102. Prove that if $f : [0, 1] \to \mathbb{R}$ is absolutely continuous and $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, then $g \circ f : [0, 1] \to \mathbb{R}$ is absolutely continuous.

Proof. (Write your solution here.) □

Problem 103. Prove that if $f : [a, b] \to \mathbb{R}$ is absolutely continuous, then for every $a < c < d < b$

$$\lim_{h \to 0} \int_c^d \left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| dx = 0\,.$$

Hint: Represent $f(x + h) - f(x)$ as an integral of the derivative and use an argument similar to that in the proof of Lemma 93 in my notes.

Proof. (Write your solution here.) □

Problem 104. Prove that if $\mu \ll \nu$ and $\mu \perp \nu$, then $\mu = 0$.

Proof. (Write your solution here.) □

Problem 105. Prove that if μ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}_n$ and

$$\lim_{r \to 0} r^{-n} \mu(B(x, r)) = 0 \quad \text{for every } x \in \mathbb{R}^n$$

then $\mu = 0$.

Proof. (Write your solution here.) □

Introduction to Problem 106: it is well known that there are homeomorphisms $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ with the property that for some sets $E \subset \mathbb{R}^n$ of measure zero, the set $\varphi(E)$ has positive measure.

Problem 106. Prove that if $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism, then there is a set $Z \subset \mathbb{R}^n$ of measure zero $|Z| = 0$ such that for each measurable set A the following implication is satisfied:

$$A \subset \mathbb{R}^n \setminus Z, |A| = 0 \implies |\varphi(A)| = 0\,.$$

Hint: $\mu(E) = |\varphi(E)|$, $E \subset \mathbb{R}^n$ defines a measure in $\mathbb{R}^n$. Use the Radon-Nikodym-Lebesgue theorem.

Proof. (Write your solution here.) □

Problem 107. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite. Suppose that F is $\mathcal{M} \times \mathcal{N}$ measurable. Prove the following inequality

$$\left(\int_Y \left(\int_X |F(x,y)|\, d\mu(x) \right)^p d\nu(y) \right)^{1/p} \leq \int_X \left(\int_Y |F(x,y)|^p \, d\nu(y) \right)^{1/p} d\mu(x).$$

Hint: On the left hand side we compute the L^p norm of the function $y \mapsto \int_X |F(x,y)|\, d\mu(x)$. An L^p function defines a functional on L^q. Hence the L^p norm equals to the supremum of certain integrals over certain functions in L^q. Use this fact in the proof of the inequality.

Proof. (Write your solution here.) □