Change of variables formula

$L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ linear map.

B - ball L(B) - ellipsoid

$$\frac{|L(B)|}{|B|} = |\det L|.$$

True also when $\det L = 0$. In the class we proved

Theorem For a measurable set $E \subset \mathbb{R}^n$ we have
$$|L(E)| = |\det L|\,|E|$$

$\Omega \subset \mathbb{R}^n$ open, $\Phi : \Omega \to \Phi(\Omega) \subset \mathbb{R}^n$ diffeomorphism.

$\Omega = \bigcup_{i=1}^{\infty} Q_i$ - union of very small cubes Q_i.

Denote the center of Q_i by x_i.

For a measurable set $E \subset \Omega$ we have (intuitively, not rigorously)

$$\Phi(x) \approx \Phi(x_i) + D\Phi(x_i)(x - x_i) \quad \text{for } x \in Q_i.$$

$$|\Phi(E\cap Q_i)| \approx |D\Phi(x_i)(E\cap Q_i)| = |\det D\Phi(x_i)|\,|E\cap Q_i|$$

$$= \int_{E\cap Q_i} |\det D\Phi(x_i)|\,dx \approx \int_{E\cap Q_i} |\det D\Phi(x)|\,dx.$$

Taking partition into smaller cubes gives a better approximation so passing to the limit yields

$$|\Phi(E)| = \int_E |\det D\Phi|.$$

Using notation $|J_\Phi| = |\det D\Phi|$ the result can be stated as

<u>Theorem</u> If $\Phi : \Omega \to \mathbb{R}^n$ is a diffeomorphism and $E \subset \Omega$ is measurable, then

$$|\Phi(E)| = \int_\Omega |J_\Phi|\,dx$$

If $A \subset \Phi(\Omega)$, then $A = \Phi(E)$, where $E = \Phi^{-1}(A)$. Thus we have

$$\int_{\mathbb{R}^n} \chi_A = |A| = |\Phi(E)| = \int_E |J_\Phi| =$$

$$\int_{\Phi^{-1}(A)} |J_\Phi| = \int_{\mathbb{R}^n} \chi_{\Phi^{-1}(A)} |J_\Phi| =$$

$$\int_{\mathbb{R}^n} (\chi_A \circ \Phi)(x) \, |J_\Phi(x)| \, dx.$$

That is

$$\int_{\mathbb{R}^n} \chi_A = \int_{\mathbb{R}^n} (\chi_A \circ \Phi)(x) |J_\Phi(x)| dx.$$

Since simple functions are linear combinations of characteristic functions, if s is a simple function on $\Phi(\Omega)$, then

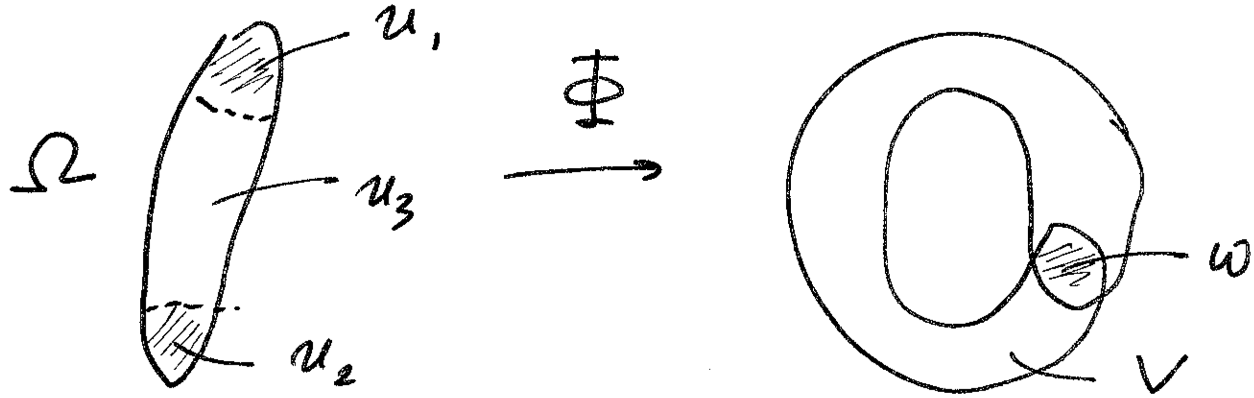
$$\int_{\mathbb{R}^n} s(y) dy = \int_{\mathbb{R}^n} (s \circ \Phi)(x) |J_\Phi(x)| dx.$$

Since measurable functions can be approximated by simple functions

$$\int_{\mathbb{R}^n} f dy = \int_{\mathbb{R}^n} (f \circ \Phi)(x) | J_\Phi(x) | dx$$

whenever $f : \Phi(\Omega) \longrightarrow [0, \infty]$ is measurable.

Suppose now that $\Phi : \Omega \to \mathbb{R}^n$ is a local diffeomorphism with an overlapping as on the picture



$$\Phi(U_1) = \Phi(U_2) = W, \quad \Phi(U_3) = V.$$

For a measurable function $f : \Phi(\Omega) \to [0, \infty]$ we have

$$\int_W f = \int_{U_1} (f \circ \Phi) |J_\Phi| ,$$

$$\int_W f = \int_{U_2} (f \circ \Phi) |J_\Phi| ,$$

$$\int_V f = \int_{U_3} (f \circ \Phi) |J_\Phi| ,$$

$$\int_W 2f + \int_V f = \int_{\underbrace{U_1 \cup U_2 \cup U_3}_{\Omega}} (f \circ \Phi) |J_\Phi|$$

$$(*) \quad \int_{\Phi(\Omega)} f(y)\, N_\Phi(y)\, dy = \int_\Omega (f \circ \Phi) |J_\Phi|$$

where $N_\Phi(y) = \# \Phi^{-1}(y)$ - the number of points in $\Phi^{-1}(y)$. The function

The function N_Φ is called the Banach indicatrix.

The equality (*) is true in a great generality that we describe next.

Theorem (Rademacher) If $f : \Omega \longrightarrow \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ open, is Lipschitz, then f is differentiable a.e.

Thus if $\Phi : \Omega \longrightarrow \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ open, is any Lipschitz map, then Φ is differentiable a.e. and hence

$$|J_\Phi| = |\det D\Phi|$$

exists a.e.

The following result is true.

Theorem If $\Phi : \Omega \longrightarrow \mathbb{R}^n$ is Lipschitz continuous and if $f : \Phi(\Omega) \longrightarrow [0, \infty]$ is measurable, then

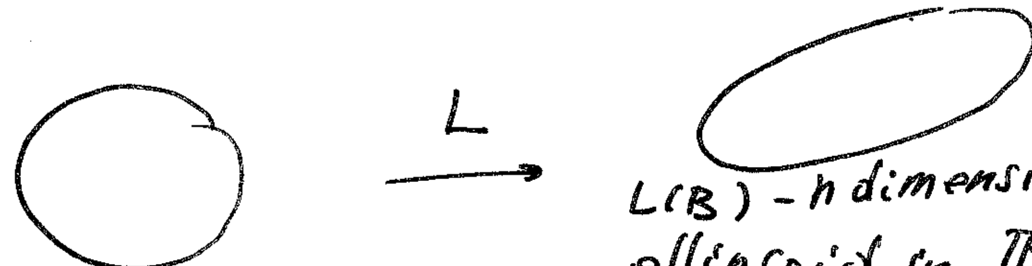
$$\int_\Omega (f \circ \Phi) |J_\Phi| \, dx = \int_{\Phi(\Omega)} f(y) \, N_\Phi(y) \, dy .$$

This result can be generalized to the case of mappings $\Phi : \Omega \longrightarrow \mathbb{R}^m$ where $m > n$.

For a linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ we have

$$|\det L| = \sqrt{\det(L^T L)} = \sqrt{\det(L L^T)} .$$

If $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $m \geq n$ is a linear map and $\ker L = 0$, then



$$\mathcal{H}^n(L(B)) = \sqrt{\det(L^T L)}\ |B| .$$

How do we integrate on parametric surfaces?

Let's recall what we learned in Calculus 3.

If $r : \mathbb{R}^2 \supset \Omega \longrightarrow M \subset \mathbb{R}^3$ is a smooth and regular parametrization of a surface

$$r(u,v) = (x(u,v), y(u,v), z(u,v)),$$

then

$$\int_M f\,dS = \int_\Omega f(r(u,v))\,|r_u \times r_v|\,dA.$$

It is easy to check that

$$|r_u \times r_v| = \sqrt{\det (Dr)^T (Dr)}$$

Thus, replacing $r : \Omega \longrightarrow \mathbb{R}^3$ by $\Phi : \Omega \longrightarrow \mathbb{R}^3$ the above formula reads as

$$\int_{\Phi(\Omega)} f\,dS = \int_\Omega (f \circ \Phi) \sqrt{\det (D\Phi)^T D\Phi}.$$

Note that Φ is one-to-one (as a parametrization of a surface) and hence $N_\Phi(y) = \# \Phi^{-1}(y) = 1$.

The above formula is a special case of the following general result.

<u>Theorem (The area formula)</u> If $\Phi : \Omega \rightarrow \mathbb{R}^m$, $\Omega \subset \mathbb{R}^n$, $m \geqslant n$ is Lipschitz and $f : \Phi(\Omega) \rightarrow [0, \infty]$ measurable, then

$$\int_{\Phi(\Omega)} f(y)\, N_\Phi(y)\, d\mathcal{H}^n(y) = \int_\Omega (f \circ \Phi)\, |J_\Phi|\, dx,$$

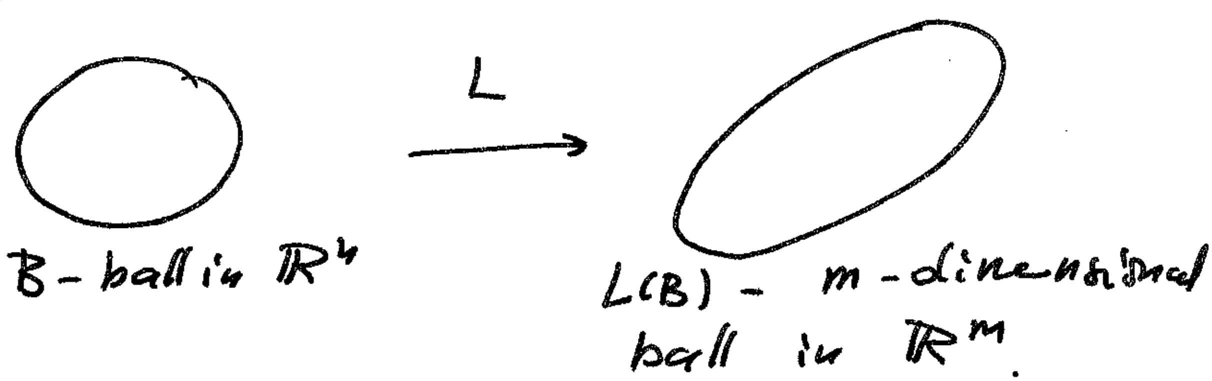
where $|J_\Phi|(x) = \sqrt{\det (D\Phi)^T D\Phi}\,(x).$

If $m=n$, this is the same result as the theorem on p. 5.

The change of variables formula can be also generalized to the case of Lipschitz mappings $\Phi: \mathbb{R}^n \supset \Omega \longrightarrow \mathbb{R}^m$, $m<n$.

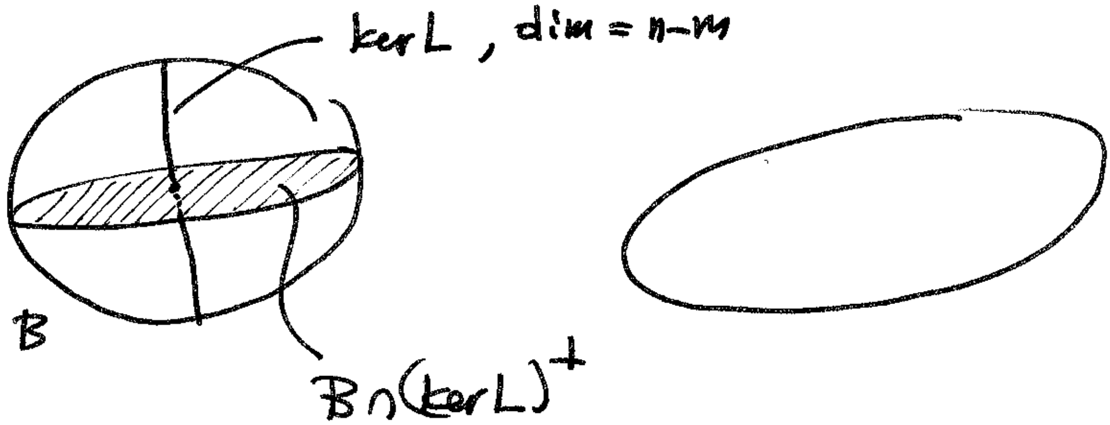
The corresponding result is known as the coarea formula which includes the Fubini theorem and the integration in spherical coordinates as special cases.

Let $L: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $m \le n$ be a linear map and rank $L = m$.



If $m<n$, then L must have a nontrivial kernel. In fact

$$\dim \ker L = n-m.$$



Let $W^m \subset \mathbb{R}^n$ be a linear subspace of dimension m such that

$$B \cap (\ker L)^\perp \subset W^m.$$

Then $B \cap (\ker L)^\perp$ is the unit ball in W^m and

$$L|_{W^m} : W^m \longrightarrow \mathbb{R}^m$$

is a non-degenerate linear map that maps the unit ball $B \cap (\ker L)^\perp$ onto $L(B)$. From what we discussed on page 1,

$$\frac{\mathcal{H}^m(L(B))}{\mathcal{H}^m(B \cap (\ker L)^\perp)} = |\det(L|_{W^m})|.$$

Using a linear algebra one can show that

$$|\det(L|_{W^m})| = \sqrt{\det(LL^T)}.$$

Thus for a linear map $L: \mathbb{R}^n \to \mathbb{R}^m$, $m \le n$ we have

$$\frac{\mathcal{H}^m(L(B))}{\mathcal{H}^n(B \cap (\ker L)^\perp)} = \sqrt{\det(LL^T)}.$$

While we discussed this formula assuming that rank $L = m$, it is true for any linear map $L: \mathbb{R}^n \to \mathbb{R}^m$.

Note that $L^{-1}(y)$ is a linear subspace of dimension $n-m$ (assuming rank $L = m$). Thus in general, we should expect that if $\Phi: \mathbb{R}^n \to \mathbb{R}^m$ is smooth or even just Lipschitz continuous, then $\Phi^{-1}(y)$ has dimension $\le n-m$ for almost all y.

This motivates the following remarkable result due to Federer.

Theorem (The co-area formula)

If $\Phi : \mathbb{R}^n \supset \Omega \longrightarrow \mathbb{R}^m$, $m \leq n$ is Lipschitz and $f : \Phi(\Omega) \longrightarrow [0, \infty]$ is measurable, then

$$\int_{\Phi(\Omega)} f(y)\, \mathcal{H}^{n-m}(\Phi^{-1}(y))\, d\mathcal{H}^m(y) = \int_\Omega (f \circ \Phi)\, |J_\Phi|(x)\, dx,$$

where

$$|J_\Phi| = \sqrt{\det (D\Phi)(D\Phi)^T}.$$

Remark Note that in the case of the area formula when $m \geq n$

$$|J_\Phi| = \sqrt{\det (D\Phi)^T (D\Phi)}.$$

Example If $f = 1$, the coarea formula reads as

$$\int_\Omega |J_\Phi| = \int_{\Phi(\Omega)} \mathcal{H}^{n-m}(\Phi^{-1}(y))\, d\mathcal{H}^m(y)$$

$$= \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(\Phi^{-1}(y))\, d\mathcal{H}^m(y)$$

because $\Phi^{-1}(y) = \emptyset$ for $y \notin \Phi(\Omega)$

<u>Example</u> If $\Phi : \Omega \longrightarrow \mathbb{R}$ is Lipschitz, then

$$(D\Phi)(D\Phi)^T = [\ \dots\ ]\begin{bmatrix} \vdots \end{bmatrix} = |\nabla \Phi|^2$$

and hence

$$|J_\Phi| = \sqrt{\det (D\Phi)(D\Phi)^T} = |\nabla \Phi|$$

Thus using notation $f = \Phi$ (more traditional for real valued functions, the formula from the previous example reads as

$$\int_\Omega |\nabla f| = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(f^{-1}(y))\, dy .$$

This formula has a beautiful geometric meaning.

The coarea formula can be generalized as follows.

<u>Theorem</u> If $\Phi : \mathbb{R}^n \supset \Omega \longrightarrow \mathbb{R}^m$, $m \leq n$ is Lipschitz and $f : \Omega \longrightarrow [0, \infty]$ is measurable, then

$$\int_{\mathbb{R}^m} \left(\int_{\Phi^{-1}(y) \cap \Omega} f(x) \, d\mathcal{H}^{n-m}(x) \right) d\mathcal{H}^m(y) =$$

$$= \int_\Omega f(x) \, |J_\Phi|(x) \, dx ,$$

where

$$|J_\Phi| = \sqrt{\det (D\Phi)(D\Phi)^T}$$

<u>Remark</u> The difference is that in the previous formula f was defined on $\Phi(\Omega)$ and now f is defined on Ω. The previous formula from p. 11 follows from this one applied to $f \circ \Phi$ in place of f. Indeed, if f is defined on $\Phi(\Omega)$, then $f \circ \Phi$ is defined on Ω so the above theorem applies to $f \circ \Phi$.

As an applocation we will show how to conclude both the Fubini theorem and integration in spherical coordinate system.

<u>Example (Fubini theorem)</u> Let $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ be the orthogonal projection. Then using either linear algebra or geometric interpretation, one can check that $|J\Phi| = 1$.

Denote the coordinates in $\mathbb{R}^n$ by
$\mathbb{R}^n = \mathbb{R}^{n-m} \times \mathbb{R}^m = \{(x,y) \mid x \in \mathbb{R}^{n-m},\ y \in \mathbb{R}^m\}$,

so $\Phi^{-1}(y) = \mathbb{R}^{n-m} \times \{y\}$. Thus if $f : \mathbb{R}^n \to \mathbb{R}$ is a function, we have that

$$\int_{\Phi^{-1}(y)} f \, d\mathcal{H}^{n-m} = \int_{\mathbb{R}^{n-m}} f(x,y) \, d\mathcal{H}^{n-m}(x).$$

Therefore, the co-area formula from p. 13 yields

$$\int_{\mathbb{R}^n} f \, d\mathcal{H}^n = \int_{\mathbb{R}^n} f |J_\Phi| \, d\mathcal{H}^n =$$

$$\int_{\mathbb{R}^m} \left(\int_{\Phi^{-1}(y)} f \, d\mathcal{H}^{n-m} \right) d\mathcal{H}^m =$$

$$\int_{\mathbb{R}^m} \int_{\mathbb{R}^{n-m}} f(x,y) \, d\mathcal{H}^{n-m}(x) \, d\mathcal{H}^m(y)$$

which is the Fubini theorem.

<u>Example (spherical coordinates)</u>

Let $\Phi : \mathbb{R}^n \to \mathbb{R}$, $\Phi(x) = |x|$.
Then $\Phi^{-1}(t) = S^{n-1}(0,t)$, $t > 0$
and $\Phi^{-1}(t) = \phi$ if $t < 0$. Moreover

$$|J_\Phi| \underset{\uparrow}{=} |\nabla \Phi| = 1 .$$

we already discussed it

Therefore the co-area formula p. 13 yields

$$\int_{\mathbb{R}^n} f \, d\mathcal{H}^n = \int_0^\infty \left(\int_{S^{n-1}(0,t)} f \, d\mathcal{H}^{n-1} \right) dt .$$