# Convergence in measure

**Definition** $f_n : X \to \mathbb{R}$ converges in measure to $f : X \to \mathbb{R}$ if

$$\forall_{\varepsilon>0} \quad \lim_{n\to\infty} \mu(\{x \mid |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

We denote the convergence in measure by $f_n \xrightarrow{\mu} f$.

**Theorem (Lebesgue)** If $\mu(X) < \infty$ and $f_n \to f$ a.e., then $f_n \xrightarrow{\mu} f$.

**Proof** Fix $\varepsilon > 0$. Let $E_i = \{|f_i(x) - f(x)| \geq \varepsilon\}$.

$\bigcup_{i=n}^\infty E_i$ is a decreasing sequence of sets of finite measure $\Longrightarrow$

$$\mu\left(\bigcup_{i=n}^\infty E_i\right) \to \mu\left(\bigcap_{n=1}^\infty \bigcup_{i=n}^\infty E_i\right) = 0.$$

To see that the measure equals zero observe that

$$x \in \bigcap_{n=1}^\infty \bigcup_{i=n}^\infty E_i \Longrightarrow \forall_n \exists_{i \geq n} \underbrace{|f_i(x) - f(x)| \geq \varepsilon}_{x \in E_i}$$

$$\Longrightarrow f_i(x) \not\to f(x)$$

so the set of such $x$ has measure zero.

Therefore

$$\mu(\{|f_n - f| \geq \varepsilon\}) = \mu(E_n) \leq \mu\left(\bigcup_{i=n}^\infty E_i\right) \to 0.$$

$\square$

**Theorem** (Riesz) If $f_n \xrightarrow{\mu} f$, then there is a subsequence $f_{n_i}$ such that $f_{n_i} \to f$ a.e.

**Remark** We allow $\mu(X) = \infty$.

**Proof** $\forall_i \quad \mu(\{|f_n - f| \geq \frac{1}{i}\}) \to 0$ so

$\forall_i \exists_{N_i} \forall_{n \geq N_i} \quad \mu(\{|f_n - f| \geq \frac{1}{i}\}) \leq \frac{1}{2^i}$.

Take $n_1$ such that $\mu(\{|f_{n_1} - f| \geq \frac{1}{1}\}) \leq \frac{1}{2^1}$

Take $n_2 > n_1$ such that $\mu(\{|f_{n_2} - f| \geq \frac{1}{2}\}) \leq \frac{1}{2^2}$

$\vdots$

$n_i > n_{i-1}$ such that $\mu(\{|f_{n_i} - f| \geq \frac{1}{i}\}) \leq \frac{1}{2^i}$.

We will prove that $f_{n_i} \to f$ a.e.

Let $F_k = X \setminus \bigcup_{i=k}^{\infty} \{|f_{n_i} - f| \geq \frac{1}{i}\}$

We will prove that

$f_{n_i} \rightrightarrows f$ uniformly on $F_k$.

$x \in F_k \Rightarrow x \in \bigcap_{i=k}^{\infty} (X \setminus \{|f_{n_i} - f| \geq \frac{1}{i}\})$

$\Rightarrow x \in \bigcap_{i=k}^{\infty} \{|f_{n_i} - f| < \frac{1}{i}\}$

$\forall_{x \in F_k} \forall_{i \geq k} \quad |f_{n_i} - f| < \frac{1}{i}$

$\sup_{F_k} |f_{n_i} - f| \leq \frac{1}{i}$ for $i \geq k$

$f_{n_i} \rightrightarrows f$ uniformly on $F_k$

$f_{n_i} \to f$ pointwise on $\bigcup_{k=1}^{\infty} F_k$ 

It remains to show that

$$\mu\Big(X \setminus \bigcup_{k=1}^{\infty} F_k\Big) = 0. \qquad (*)$$

Recall that

$$F_n = X \setminus \bigcup_{i=n}^{\infty} \{|f_{n_i} - f| \geq \tfrac{1}{i}\}$$

$$\mu(X \setminus F_n) \leq \sum_{i=n}^{\infty} \mu\big(\{|f_{n_i} - f| \geq \tfrac{1}{i}\}\big)$$

$$\leq \sum_{i=n}^{\infty} \frac{1}{2^i} = \frac{1}{2^{n-1}}$$

$$\mu\Big(X \setminus \bigcup_{k=1}^{\infty} F_k\Big) \underset{\text{for any } n}{\leq} \mu(X \setminus F_n) \leq \frac{1}{2^{n-1}}$$

and (*) follows.

<u>Lemma</u> If $f_n, f \in L^1(\mu)$ and $\int_X |f_n - f|\, d\mu \to 0$ as $n \to \infty$ then $f_n \xrightarrow{\mu} f$.

<u>Proof</u> For any $\varepsilon > 0$

$$\varepsilon \chi_{\{|f_n - f| \geq \varepsilon\}} \leq |f - f_n|$$

$$\varepsilon \mu(\{|f_n - f| \geq \varepsilon\}) \leq \int_X |f - f_n| \to 0$$

$$\mu(\{|f_n - f| \geq \varepsilon\}) \to 0$$

$$f_n \xrightarrow{\mu} f$$

Absolute continuity of the integral

<u>Theorem</u> Let $f \in L^1(\mu)$. Then for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$\mu(E) < \delta \implies \int_E |f| \, d\mu < \varepsilon.$$

<u>Proof</u> Suppose to the contrary that

$$\exists \varepsilon > 0 \; \forall \delta > 0 \; \exists E \in \mathfrak{M} \; \mu(E) < \delta \wedge \int_E |f| \geq \varepsilon$$

Fix such $\varepsilon > 0$. For $\delta = \frac{1}{2^n}$ we find $E_n \in \mathfrak{M}$ satisfying

$$\mu(E_n) < \frac{1}{2^n} \quad \wedge \quad \int_{E_n} |f| \geq \varepsilon$$

The family of sets

$$A_k = \bigcup_{n=k}^{\infty} E_n \text{ is decreasing}$$

and

$$\forall n \quad \mu\left(\bigcap_{k=1}^{\infty} A_k\right) \leq \mu(A_n) \leq \sum_{i=n}^{\infty} \mu(E_i)$$

$$\leq \sum_{i=n}^{\infty} \frac{1}{2^i} = \frac{1}{2^{n-1}}$$

so

$$\mu\left(\bigcap_{k=1}^{\infty} A_k\right) = 0$$

We proved that

$$\varphi(E) := \int_E |f| \, d\mu$$

is a measure on $\mathfrak{M}$. Since

$\varphi(A_1) = \int_{A_1} |f| \leq \int_X |f| < \infty$ we have

$$\varphi(A_n) \longrightarrow \varphi\left(\bigcap_{k=1}^\infty A_k\right) = \int_{\bigcap_{k=1}^\infty A_k} |f| = 0$$

$\bigcap_{k=1}^\infty A_k \leftarrow$ measure zero.

On the other hand

$$\varphi(A_n) \geq \underbrace{\varphi(E_n)}_{\text{subset of } A_n} = \int_{E_n} |f| \geq \varepsilon.$$

which is a contradiction.

<u>Definition</u> We say that a family of functions $\mathcal{F} \subset L^1(\mu)$ is uniformly integrable if (equiintegrable)

$\forall_{\varepsilon>0} \exists_{\delta>0} \forall_{f\in\mathcal{F}} \forall_{E\in\mathfrak{M}}$

$$\mu(E) < \delta \Longrightarrow \int_E |f| \, d\mu < \varepsilon.$$

We just proved that if $f \in L^1(\mu)$, then $\{f\}$ is uniformly integrable,

**Theorem** If $f_n, f \in L^1(\mu)$ and $\int_X |f_n - f| d\mu \to 0$, then the family $\{f_n\}_{n=1}^{\infty}$ is uniformly integrable.

**Proof** Given $\varepsilon > 0$ let $n_0$ be such that

$$\int_X |f - f_n| d\mu < \frac{\varepsilon}{2} \quad \text{for } n > n_0.$$

~~Applying the~~ theorem about absolute continuity of ~~the~~ integral to

$$|f| + |f_1| + \dots + |f_{n_0}| \in L^1(\mu)$$

we find $\delta > 0$ such that

$$\mu(E) < \delta \Rightarrow \int_X |f| + |f_1| + \dots + |f_{n_0}| d\mu < \varepsilon/2.$$

If $\mu(E) < \delta$ then for any $n$ we have

① $n \le n_0 \Longrightarrow \int_E |f_n| d\mu < \varepsilon$

② $n > n_0 \Longrightarrow$

$$\int_E |f_n| \le \int_E |f_n - f| + \int_E |f| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \square$$

We proved that if ~~$f_n$~~
$f_n, f \in L^1$ and $\int_X |f - f_n| \to 0$
then
$$f_n \xrightarrow{\mu} f$$
and $\{f_n\}_{n=1}^{\infty}$ is uniformly integrable.

<u>Theorem (Vitali)</u> If $\mu(X) < \infty$
and $f_n, f \in L^1(\mu)$, then
$$\int_X |f_n - f| \, d\mu \to 0$$
(and hence $\int_X |f_n| \to \int_X |f|$ )
if and only if
$f_n \xrightarrow{\mu} f$ and $\{f_n\}_{n=1}^{\infty}$ is uniformly integrable.

<u>Proof</u> We already proved the implication $\Longrightarrow$. Proof of $\Longleftarrow$.
Suppose to the contrary that there is $\varepsilon > 0$ and a subsequence such that
$$(*) \quad \int_X |f_{n_k} - f| \, d\mu \geq \varepsilon.$$
Using Riesz's theorem we may further assume that $f_{n_k} \to f$ a.e.

For this $\varepsilon > 0$
~~Fix $\varepsilon > 0$~~ ~~and~~ let $\delta > 0$ be such that

$$\mu(E) < \delta \Rightarrow \int_E |f| + |f_n| < \varepsilon/2 \text{ for all } n.$$

According to Egorov's theorem there is $F \in \mathfrak{M}$ such that

$\mu(X \setminus F) < \delta$ and $f_{n_k} \rightrightarrows f$ uniformly on $F$.

Therefore,

$$\int_X |f_{n_k} - f| = \int_{\cancel{F}} |f_{n_k} - f| + \underbrace{\int_{X \setminus F}}_{\mu(X\setminus F) < \delta} \underbrace{|f_{n_k} - f|}_{\leq |f_{n_k}| + |f|}$$

$$\leq \int_F \underbrace{|f_{n_k} - f|}_{\rightrightarrows 0} + \frac{\varepsilon}{2}.$$

$\leq \varepsilon$ for sufficiently large $k$.

This contradicts (*).