Doubling measures

If $\sigma > 0$ and $B = B(x, r)$, we write $\sigma B = B(x, \sigma r)$. The same notation applies to closed balls.

Definition We say that a measure μ on a metric space is doubling if all Borel sets are measurable and there is a constant $C_d > 0$ (doubling constant) such that all balls B in X satisfy

$$0 < \mu(B) \leq C_d \, \mu(2B) < \infty$$

Example $\mathcal{L}_n$ on $\mathbb{R}^n$ is doubling; $C_d = 2^n$

Definition A metric space X is said to be metric-doubling if there is a constant $M \geq 1$ such that every ball can be covered by no more than M balls of half the radius.

Formally:

$$\forall x \in X \; \exists r > 0 \; \exists k \leq M \; \exists x_1, \dots, x_k \in X$$

$$B(x, r) \subset \bigcup_{i=1}^{k} B\left(x_i, \frac{r}{2}\right)$$

Examples

- $\mathbb{R}^n$ is metric - doubling
- X - metric doubling, $Y \subset X \Rightarrow Y$ - metric doubling
- X - infinite discrete space $\Rightarrow$ not metric - doubling

Proposition Every complete metric - doubling space is boundedly compact (or proper), meaning that bounded and closed sets are compact.

Remark Being boundedly compact is a stronger condition than being locally compact: (1) $\mathbb{R}^n \setminus \{0\}$ is locally compact, but not boundedly compact; (2) If X is an infinite discrete space, then X is locally compact but not boundedly compact.

Proof Recall that a metric space is compact iff it is complete and totally bounded, meaning that for any $\varepsilon > 0$ it can be covered by finitely many balls of radius ε.

Let $E \subset X$ be bounded and closed. Clearly, E is a complete metric space so it remains to show that it is totally bounded.

Clearly, $E \subset B(x, R)$ for some $x \in X$ and $R > 0$. Therefore

- It can be covered by $\leq M$ balls of radius $\frac{R}{2}$
- It can be covered by $\leq M^2$ balls of radius $\frac{R}{4}$

$\vdots$

- It can be covered by $\leq M^k$ balls of radius $\frac{R}{2^k}$.

If $R/2^k \leq \varepsilon$, then E can be covered by $\leq M^k$ balls of radius ε. $\square$

In fact we proved

Proposition Every bounded set in a metric-doubling space X is totally bounded and hence X is separable.

I leave details ~~to~~ as an exercise.

Proposition If μ is a doubling measure on a metric space X, then X is metric-doubling.

<u>Proof</u> We will prove that X is metric-doubling with $M = C_d^4$ where C_d is the doubling constant of the measure i.e.,

$$0 < \mu(2B) \leq C_d\, \mu(B).$$

Let $\{x_i\}_{i \in I} \subset B(x, r)$ be a maximal $\frac{r}{2}$-separated set i.e.,

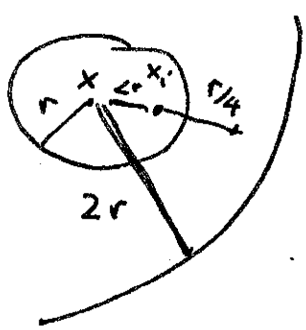
(1) $\forall i, j \in I \quad d(x_i, x_j) \geq \frac{r}{2}$

(2) $\forall y \in B(x, r) \; \exists\, i \in I \quad d(x_i, y) < \frac{r}{2}$

It follows from (2) that $B(x,r) \subset \bigcup_{i \in I} B(x_i, \frac{r}{2})$

So it remains to show that the cardinality of I satisfies $\# I \leq C_d^4$.

The balls $B(x_i, \frac{r}{4})$ are pairwise disjoint by (1) and $B(x_i, \frac{r}{4}) \subset B(x, 2r) \subset B(x_i, 4r)$



(in fact $B(x, 2r) \subset B(x_i, 3r)$ but we don't care).

We have

$$\mu(B(x_i,4r)) \le C_d\,\mu(B(x_i,2r)) \le \dots \le C_d^4\,\mu(B(x_i,\tfrac{r}{4}))$$

$$\mu(B(x_i,\tfrac{r}{4})) \ge C_d^{-4}\,\mu(B(x_i,4r))$$

$$\mu(B(x,2r)) \ge \sum_{i\in I}\mu(B(x_i,\tfrac{r}{4}))$$

$$\ge C_d^{-4}\sum_{i\in I}\mu(B(x_i,4r)) \ge C_d^{-4}\,\#I\,\mu(B(x,2r))$$

$$C_d^{-4}\,\#I \le 1, \quad \#I \le C_d^4. \qquad \square$$

Theorem (Volberg–Konyagin) If X is a complete metric-doubling space, then there is a doubling measure on X.

We will not prove it.

Corollary If $X \subset \mathbb{R}^n$ is closed, then there is a doubling measure on X.

Theorem Let μ be a doubling measure on X. Then there is a constant $C > 0$ such that

$$\frac{\mu(B(x,r))}{\mu(B(x_0,r_0))} \ge C\left(\frac{r}{r_0}\right)^s$$

whenever $x \in B(x_0,r_0)$, $r \le r_0$ and $s = \log C_d / \log 2$

<u>Remark</u> If $\mu = \mathcal{L}_n$, then $C_d = 2^n$ and hence $s = n$. In that case the theorem yields

$$\frac{|B(x,r)|}{|B(0,1)|} \geqslant c r^n, \quad x \in B(0,1),\ r \le 1.$$

This example shows that the exponent s in the theorem is sharp, meaning that it cannot be replaced by a better one.

<u>Proof.</u> Let $x \in B(x_0, r_0)$, $r \le r_0$. Let k be the largest integer such that $2^{k-1} r \le 2 r_0$ so $2^k r > 2 r_0$ and hence $B(x_0, r_0) \subset B(x, 2^k r)$ and we have

$$\mu(B(x_0,r_0)) \le \mu(B(x,2^k r)) \le C_d^{\,k}\, \mu(B(x,r))$$

$$\frac{\mu(B(x,r))}{\mu(B(x_0,r_0))} \geqslant C_d^{\,-k}. \qquad (*)$$

Since $2^{k-1} r \le 2 r_0$, $\frac{r}{r_0} \le 4\, 2^{-k}$,

$$\left(\frac{r}{r_0}\right)^s = \left(\frac{r}{r_0}\right)^{\log C_d/\log 2} \le \underbrace{\left(4\right)^{\log C_d/\log 2}}_{A} \left(2^{-k}\right)^{\log C_d/\log 2}$$

$$= A\, C_d^{\,-k}.$$

This estimate and $(*)$ give the result.

5r - covering lemma

Recall that if $B = B(x,r)$, then $5B = B(x,5r)$

Theorem (5r - covering lemma)

Let $\mathcal{F}$ be a family of balls in a metric space X such that

$$\sup_{B\in\mathcal{F}} \operatorname{diam} B < \infty .$$

Then there is a subfamily $\mathcal{F}' \subset \mathcal{F}$ of pairwise disjoint balls such that

$$\forall_{B\in\mathcal{F}} \exists_{B'\in\mathcal{F}'} \quad B\cap B' \neq \emptyset \wedge B \subset 5B' .$$

In particular

$$\bigcup_{B\in\mathcal{F}} B \subset \bigcup_{B'\in\mathcal{F}'} 5B' .$$

Balls in $\mathcal{F}$ can be all open or all closed. Proof remains the same.

Remark If in addition X is separable, the family $\mathcal{F}'$ is countable and we can arrange it as a sequence (possibly finite) $\mathcal{F}' = \{B_i\}_{i=1}^{\infty}$ so

$$\bigcup_{B\in\mathcal{F}} B \subset \bigcup_{i=1}^{\infty} 5B_i .$$

Proof Let $R = \sup_{B \in \mathcal{F}} \operatorname{diam} B < \infty$

and define

$$\mathcal{F}_j = \left\{ B \in \mathcal{F} \;\middle|\; \frac{R}{2^j} < \operatorname{diam} B \leq \frac{R}{2^{j-1}} \right\}$$

Clearly $\mathcal{F} = \bigcup_{j=1}^{\infty} \mathcal{F}_j$ is a decomposition of the family $\mathcal{F}$ into disjoint subfamilies according to diameters of the balls.

We will construct subfamilies $\mathcal{F}_j' \subset \mathcal{F}_j$ by induction.

① $\mathcal{F}_1' \subset \mathcal{F}_1$ is any maximal subfamily of pairwise disjoint balls.

Maximal, means that $\mathcal{F}_1'$ is not contained in any larger family of pairwise disjoint balls. That is

$$B \in \mathcal{F}_1 \setminus \mathcal{F}_1' \Rightarrow B \cap B' \neq \emptyset \text{ for some } B' \in \mathcal{F}_1'.$$

Existence of such a maximal family is a consequence of the axiom of choice (Hausdorff maximality principle / Zorn's lemma)

2 Suppose that families
$$\mathcal{F}_1' \subset \mathcal{F}_1, \ldots, \mathcal{F}_{j-1}' \subset \mathcal{F}_{j-1}$$
have already been selected. Now we shall define $\mathcal{F}_j' \subset \mathcal{F}_j$.

Let
$$\widetilde{\mathcal{F}}_j = \{ B \in \mathcal{F}_j \mid B \cap B' = \phi \text{ for all } B' \in \bigcup_{i=1}^{j-1} \mathcal{F}_i' \}$$
That is we only consider balls in $\mathcal{F}_j$ that are disjoint with the already selected balls: $\bigcup_{i=1}^{j-1} \mathcal{F}_i'$.

Now we define

$\mathcal{F}_j' \subset \widetilde{\mathcal{F}}_j$ to be a maximal family of pairwise disjoint balls.

Clearly, balls in $\mathcal{F}_j'$ do not intersect with balls in $\mathcal{F}_1', \ldots, \mathcal{F}_{j-1}'$. Thus we always add balls that are disjoint between themselves and from what was already selected. That implies that balls in
$$\mathcal{F}' = \bigcup_{j=1}^{\infty} \mathcal{F}_j'$$
are pairwise disjoint.

It remains to show that

$$\forall_{B \in \mathcal{F}} \exists_{B' \in \mathcal{F}'} \quad B \cap B' \neq \phi \ \wedge \ B \subset 5B'.$$

Fix any $B \in \mathcal{F}$. Then $B \in \mathcal{F}_j$ for some j. Observe that there is $B' \in \bigcup_{i=1}^{j} \mathcal{F}_i'$ such that $B' \cap B \neq \phi$.

Indeed, if no such ball exists, we could add B' to the family $\mathcal{F}_j'$ making it larger and contradict maximality of $\mathcal{F}_j'$.

It remains to show that $B \subset 5B'$.

We have

$$\operatorname{diam} B \leq \frac{R}{2^{j-1}} = 2 \cdot \frac{R}{2^j} \leq 2 \operatorname{diam} B'$$

$B \in \mathcal{F}_j$

$B' \in \bigcup_{i=1}^{j} \mathcal{F}_i' \subset \bigcup_{i=1}^{j} \mathcal{F}_i$

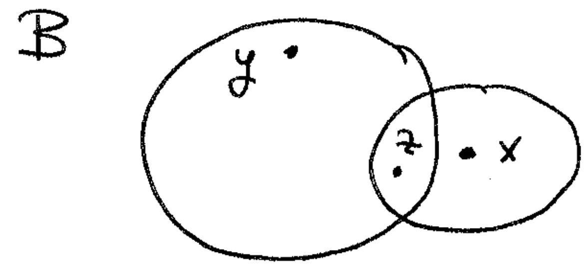
$B' \in \mathcal{F}_i$ some i

$\operatorname{diam} B' > \frac{R}{2^i} \geq \frac{R}{2^j}$

We have

$B \cap B' \neq \phi$, $\operatorname{diam} B \leq 2 \operatorname{diam} B'$ so $B \subset 5B'$.

Indeed, it follows from the triangle inequality



x - center of B'

$B' = B(x, r)$

$\operatorname{diam} B' \le 2r$

$z \in B' \cap B$

For any $y \in B$ we have

$$d(x,y) \le d(x,z) + d(z,y) \overset{(*)}{<} r + \operatorname{diam} B$$

$$\le r + 2\underbrace{\operatorname{diam} B'}_{\le 2r} \le 5r$$

$$y \in B(x, 5r) = 5B', \quad B \subset 5B'$$

(*) - this inequality is under the assumption that all balls are open. If they are closed, then we have $\le$ so $d(x,y) \le 5r$ and $y \in \overline{B}(x, 5r) = 5B'$, $B \subset 5B'$. □

Definition Let $\mathcal{F}$ be a family of balls in a metric space X. We say that $\mathcal{F}$ is a fine covering or a Vitali covering of $A \subset X$ if

$$\forall x \in A \ \forall \varepsilon > 0 \ \exists B \in \mathcal{F} \quad x \in B \subset B(x, \varepsilon)$$

That is, every point in A is covered by arbitrarily small balls in $\mathcal{F}$.

<u>Corollary</u> Let $\mathcal{F}$ be a family of <u>closed</u> balls in X, $\sup_{B\in\mathcal{F}} \operatorname{diam} B < \infty$, that is a Vitali covering of $A \subset X$. If $\mathcal{F}' \subset \mathcal{F}$ is a subfamily as in the $5r$-covering lemma, then for any $B_1', \dots B_N' \in \mathcal{F}'$ we have

$$A \subset \bigcup_{j=1}^{N} B_j' \cup \bigcup_{B' \in \mathcal{F}' \setminus \{B_1', \dots, B_N'\}} 5B' .$$

<u>Proof</u> Let $x \in A \setminus \bigcup_{j=1}^{N} B_j'$. Since the balls are closed, there is $\varepsilon > 0$ such that

$$B(x,\varepsilon) \cap B_j' = \phi \quad \text{for all } j = 1, 2, \dots, N.$$

Let $B \in \mathcal{F}$ be such that

$$x \in B \subset B(x,\varepsilon) \quad \text{so} \quad B \cap B_j' = \phi, \ j = 1, 2, \dots, N.$$

According to the $5r$-covering lemma, there is $B' \in \mathcal{F}'$ such that

$$B \cap B' \neq \phi \quad \text{and} \quad x \in B \subset 5B'.$$

Since $B \cap B_j' = \phi$, $B' \neq B_j$ for $j = 1, 2, \dots N$ i.e.

$$B' \in \mathcal{F}' \setminus \{B_1', \dots, B_N'\}$$

and the result follows. $\square$

Vitali covering theorem

Let us recall an exercise that we solved in class

Proposition If $\mathcal{U} \subset \mathbb{R}^n$ is open, then there is a family $\{B_i\}_{i=1}^\infty$ of pairwise disjoint closed balls $B_i \subset \mathcal{U}$ such that

$$|\mathcal{U} \setminus \bigcup_{i=1}^\infty B_i| = 0.$$

The proof that we discussed heavily relied on the Euclidean structure of $\mathbb{R}^n$. Namely, we used the fact that any open set can be represented as the union of cubes with pairwise disjoint interiors. Therefore, the next result is rather surprising since it holds in abstract measure spaces with unknown geometry where the notion of cubes makes no sense.

Theorem (Vitali covering theorem)
Let μ be a doubling measure on X and $A \subset X$ be a measurable set. Let $\mathcal{F}$ be a family of closed balls that cover A in the Vitali sense. Then there is a subfamily $\mathcal{F}' \subset \mathcal{F}$ of pairwise disjoint balls such that

$$(*) \qquad \mu\left(A \setminus \bigcup_{B \in \mathcal{F}'} B\right) = 0.$$

Proof If we remove from $\mathcal{F}$ all balls with radii > 1, the remaining balls will still form a Vitali covering of A so without loss of generality we may assume that

$$\forall_{B \in \mathcal{F}} \quad \operatorname{diam} B \leq 2.$$

According to the $5r$-covering lemma, there is a subfamily $\mathcal{F}' \subset \mathcal{F}$ of pairwise disjoint balls such that

$$A \subset \bigcup_{B \in \mathcal{F}} B \subset \bigcup_{B \in \mathcal{F}'} 5B.$$

We will prove that the family $\mathcal{F}'$ satisfies $(*)$.

To this end it suffices to prove that for any ball Q of radius ≥ 1,

$$\mu\Big(Q \cap \big(A \setminus \bigcup_{B \in \mathcal{F}'} B\big)\Big) = 0.$$

As we proved, every metric-doubling space is separable so the family $\mathcal{F}'$ is countable and in particular

$$\{B \in \mathcal{F}' \mid B \subset 100Q\} = \{B_i\}_{i=1}^{\infty}$$

can be arranged as a sequence.

Note that the doubling condition yields

$$\sum_{i=1}^{\infty} \mu(5B_i) \le c \sum_{i=1}^{\infty} \mu(B_i) \le C\mu(100Q) < \infty,$$

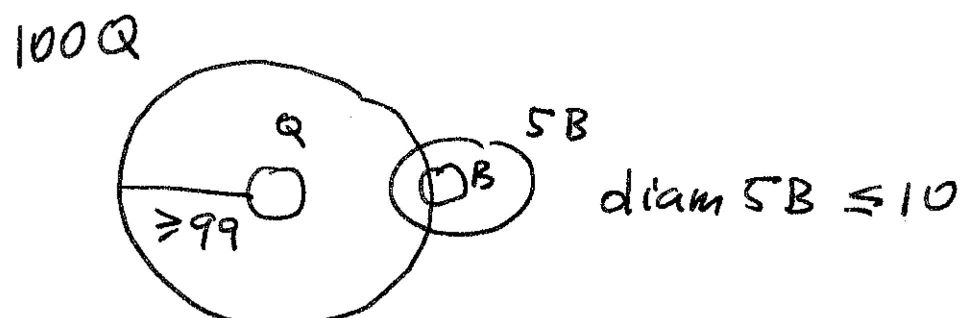
so

$$\mu\left(\bigcup_{i=N+1}^{\infty} 5B_i\right) \le \sum_{i=N+1}^{\infty} \mu(5B_i) \xrightarrow{N\to\infty} 0.$$

According to the corollary of the $5r$-covering lemma (this is where we use the assumption that the balls are closed)

$$Q \cap A \subset \bigcup_{j=1}^{N} B_j \cup \bigcup_{B \in \underbrace{\mathcal{F}' \setminus \{B_1, \dots, B_N\}}_{(\star)}} 5B.$$

If $B \in \mathcal{F}'$ but $B \not\subset 100Q$, then $5B \cap (Q \cap A) = \emptyset$ due to diameter estimates



So we can remove from $(\star)$ all balls $B \not\subset 100Q$ and the remaining balls i.e. $B_{N+1}, B_{N+2}, \dots$ will satisfy

$$Q \cap A \subset \bigcup_{j=1}^{N} B_j \cup \bigcup_{j=N+1}^{\infty} 5B_j$$

so

$$(Q \cap A) \setminus \bigcup_{j=1}^{\infty} B_j \subset (Q \cap A) \setminus \bigcup_{j=1}^{N} B_j \subset \bigcup_{j=N+1}^{\infty} 5B_j$$

$$\mu\Big((Q \cap A) \setminus \bigcup_{j=1}^{\infty} B_j\Big) \le \mu\Big(\bigcup_{j=N+1}^{\infty} 5B_j\Big) \xrightarrow{N\to\infty} 0$$

Since the measure on the left hand side does not depend on N

$$\mu\Big(Q \cap \big(A \setminus \bigcup_{j=1}^{\infty} B_j\big)\Big) = 0$$

and it suffices to remember that balls B_i belong to $\mathcal{F}$!

The proof is complete.