## Analysis 1: Exam 1

Due day: Sunday November 1, 11:59pm

NAME:

You can use any result that is **proved** in my notes without providing a proof + you can use the Fubini theorem (that is missing in my notes). All other results have to be proved. If you use a homework problem you need to provide a proof. This is an open notes+internet exam. However, you are not allowed to communicate with anyone.

**Problem 1.** Prove directly from a definition that if $f, g$ are Lebesgue measurable on $\mathbb{R}^n$, then $F(x, y) = |f(x-y)g(y)|$ is Lebesgue measurable on $\mathbb{R}^{2n}$.

*Proof.* (Write your solution here.) $\square$

**Problem 2.** Prove that if $f, f_k \in L^1(\mu)$, $f_k \to f$ a.e. and $\|f\|_1 \to \|f\|_1$, then $\|f_k - f\|_1 \to 0$.
**Hint:** *Mimic the proof of the dominated convergence theorem.*

*Proof.* (Write your solution here.) $\square$

**Problem 3.** Prove that if

$$\int_{\mathbb{R}^n} \big|\log|f_k|\big|\, dx \to 0, \quad \text{as } k \to \infty,$$

then there is a subsequence $f_{k_i}$ such that $|f_{k_i}| \to 1$ a.e.

*Proof.* (Write your solution here.) $\square$

**Problem 4.** Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then

$$\|f\|_2 \leq \liminf_{m\to\infty} \left( \frac{1}{m} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{m}\right)^2 \right)^{1/2}.$$

(**Hint:** Use Fatou.)

*Proof.* (Write your solution here.) $\square$

**Problem 5.** Prove directly from the definition of the Lebesgue measure and the definition of the integral that if $u \in L^1(\mathbb{R}^n)$ (with respect to the Lebesgue measure), then for every $t > 0$

$$\int_{\mathbb{R}^n} u(x) = t^{-n} \int_{\mathbb{R}^n} u(x/t)\, dx.$$

*Proof.* (Write your solution here.) $\square$

**Problem 6.** Prove (you can use Problem 5) that

$$\int_{|x|\geq r} \frac{dx}{|x|^{n+1}} = \frac{C(n)}{r},$$

where $C(n)$ is a constant that depends on $n$ only.
(Remember that you cannot use integration in the spherical coordinate system since it was not covered in the class yet.)

*Proof.* (Write your solution here.) $\square$

**Problem 7.** Let $X$ be a metric space and $0 < s < \infty$. Prove that if $E \subset X$, then there is a decreasing sequence of open sets $U_1 \supset U_2 \supset \ldots \supset E$ such that

$$E \subset \tilde{E} := \bigcap_{i=1}^{\infty} U_i \quad \text{and} \quad \mathcal{H}^s(E) = \mathcal{H}^s(\tilde{E}).$$

(Note that we **do not** assume measurability of $E$).

*Proof.* (Write your solution here.) $\square$

**Problem 8.** Prove that

$$\int_{\mathbb{R}^n} |f(x)|^{2020}\, dx = 2020 \int_0^\infty t^{2019} |\{x : |f(x)| > t\}|\, dt.$$

*Proof.* (Write your solution here.) $\square$

**Problem 9.** Let $\tau_y f(x) = f(x + y)$. Prove that if $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}^n)$, then

$$\lim_{|y|\to\infty} \|\tau_y f + f\|_p = 2^{1/p} \|f\|_p.$$

**Hint.** *Compare with Lemma 93 in my notes.*

*Proof.* (Write your solution here.) $\square$

The integral average is often denoted as a barred integral:

$$\fint_X f\, d\mu = \mu(X)^{-1} \int_X f\, d\mu.$$

**Problem 10.** Prove that if $0 < \mu(X) < \infty$, then $\left(\fint_X |f|^p\, d\mu\right)^{1/p} \leq \left(\fint_X |f|^q\, d\mu\right)^{1/q}$ for all $0 < p < q < \infty$.

*Proof.* (Write your solution here.) $\square$