Problem 1 Prove that if $f \in L^1(\mathbb{R}^n)$ or if $f \geq 0$ is measurable, then for any $x \in \mathbb{R}^n$ we have

$$\int_{\mathbb{R}^n} f(z)\,dz = \int_{\mathbb{R}^n} f(-z)\,dz = \int_{\mathbb{R}^n} f(z+x)\,dz$$

This follows from the definition of the integral and the fact that if $A \subset \mathbb{R}^n$ is measurable, then

$$|A| = |A+x|, \quad |A| = |-A|$$

Details - exercise for you.

Later we will prove a general change of variables formula so that Problem 1 will be a special case of it.

In particular

$$\int_{\mathbb{R}^n} f(z)\,dz = \int_{\mathbb{R}^n} f(x-z)\,dz$$

Indeed,

$$(1) \quad \int_{\mathbb{R}^n} f(z)\,dz = \int_{\mathbb{R}^n} f(x+z)\,dz \overset{z \mapsto -z}{=} \int_{\mathbb{R}^n} f(x-z)\,dz.$$

Problem 2 Let $\alpha > 0$. Prove that

$$I = \int_{B^n(0,1)} \frac{dx}{|x|^\alpha} < \infty \text{ iff } \alpha < n.$$

Proof. $B^n(0,1) = \bigcup_{i=0}^{\infty} \{2^{-(i+1)} \le |x| < 2^{-i}\}$

$$2^{-(i+1)} \le |x| < 2^{-i} \implies 2^{i\alpha} < \frac{1}{|x|^\alpha} \le 2^{(i+1)\alpha}$$

$$|\{2^{-(i+1)} \le |x| < 2^{-i}\}| = \omega_n (2^{-i})^n - \omega_n (2^{-(i+1)})^n$$

$$= \omega_n (2^{-in} - 2^{-in} 2^{-n}) = \underbrace{\omega_n (1-2^{-n})}_{c_n} 2^{-in}$$

$$\sum_{i=0}^{\infty} 2^{i\alpha} c_n 2^{-in} < I \le \sum_{i=0}^{\infty} 2^{(i+1)\alpha} c_n 2^{-in}$$

$$c_n \sum_{i=0}^{\infty} 2^{-i(n-\alpha)} < I \le c_n 2^\alpha \sum_{i=0}^{\infty} 2^{-i(n-\alpha)}$$

and it remains to observe that the series converges if and only if $n - \alpha > 0$ i.e. $\alpha < n$. $\square$

With a similar argument we can prove that for any $\varepsilon > 0$

$$\int_{B^n(0,\varepsilon)} \frac{dz}{|z|^\alpha} < \infty \text{ iff } \alpha < n \tag{2}$$

Problem 3 Let $\varepsilon > 0$ and $\alpha > 0$, and $x \in \mathbb{R}^n$.
Prove that

$$\int_{B^n(x,\varepsilon)} \frac{dz}{|x-z|^\alpha} < \infty \text{ iff } \alpha < n.$$

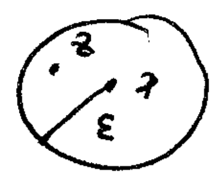
We can mimic the proof of (2) by taking balls centered at x. Instead, we will show how to use a change of variables and reduce Problem 3 to (2).

We have

$$\int_{B^n(0,\varepsilon)} \frac{dz}{|z|^\alpha} = \int_{\mathbb{R}^n} \frac{\chi_{B(0,\varepsilon)}(z)}{|z|^\alpha}\, dz$$

$$\overset{(1)}{=} \int_{\mathbb{R}^n} \frac{\chi_{B(0,\varepsilon)}(x-z)}{|x-z|^\alpha}\, dz = \heartsuit$$

$$x - z \in B(0,\varepsilon) \iff z \in B(x,\varepsilon)$$



$$\heartsuit = \int_{\mathbb{R}^n} \frac{\chi_{B(x,\varepsilon)}(z)}{|x-z|^\alpha}\, dz$$

$$= \int_{B(x,\varepsilon)} \frac{dz}{|x-z|^\alpha}.$$

so Problem 3 follows from (2).

Problem 4 Prove that if $r > 0$, then

$$\int_{\{|x| \geq r\}} \frac{dx}{|x|^{n+1}} = \frac{C(n)}{r}.$$

That can be proved by a similar method to the one used in Problem 2.

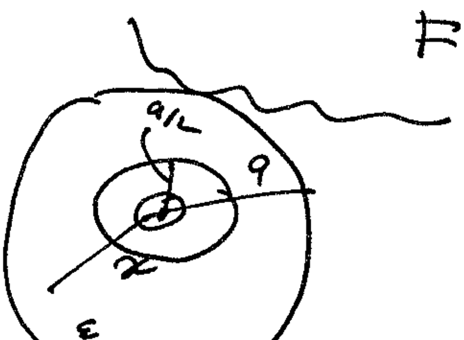
Consider now a closed set $F \subset \mathbb{R}^n$ and the distance function

$$\delta(z) = \operatorname{dist}(z, F)$$

Problem 5 Prove that if $x \notin F$ then

$$\int_{B(x,1)} \frac{\delta(z)}{|x-z|^{n+1}}\, dz = \infty.$$

Indeed, if $a = \operatorname{dist}(x, F)$ and $\varepsilon < \frac{a}{2}$



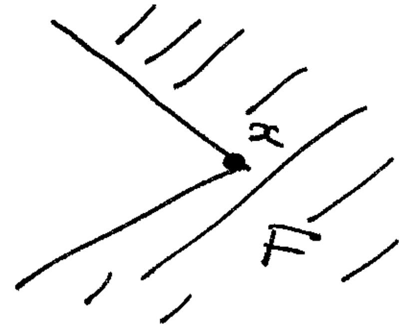
then $\delta(z) > \frac{a}{2}$ for $z \in B(x,\varepsilon)$ and hence

$$\int_{B(x,\varepsilon)} \frac{\delta(z)}{|x-z|^{n+1}} \geq \frac{a}{2} \int_{B(x,\varepsilon)} \frac{dz}{|x-z|^{n+1}} = \infty$$

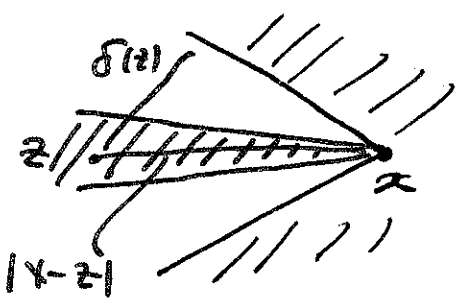
by Problem 3.

Problem 6 Let F be the angle as on the picture and let x be the vertex. Prove that

$$\int_{B(x,1)} \frac{\delta(z)}{|x-z|^{n+1}}\, dz = \infty$$



Proof



For $z \in$ we have

$$|x-z| \approx \delta(z)$$

i.e.

$$\delta(z) \leq |x-z| \leq c\,\delta(z)$$

Therefore for such z

$$\frac{\delta(z)}{|x-z|^{n+1}} \approx \frac{|x-z|}{|x-z|^{n+1}} = \frac{1}{|x-z|^{n}}$$

and hence

$$\int_{B(x,1)} \frac{\delta(z)}{|x-z|^{n+1}}\, dz \geq \int \frac{\delta(z)}{|x-z|^{n+1}}\, dz$$

$$\approx \int \frac{dz}{|x-z|^{n}} = \infty$$

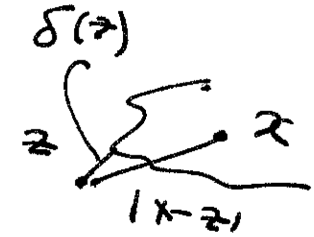
because

$$\int \frac{dz}{|x-z|^{n}} = \infty$$

Problem 7 If $x \in \operatorname{Int} F$, then
$$\int_{B(x,1)} \frac{\delta(z)}{|x-z|^{n+1}}\, dz < \infty,$$

Indeed, $\delta(z) = 0$ for $z \in B(x,\varepsilon) \subset F$
$$\int_{B(x,1)} \frac{\delta(z)}{|x-z|^{n+1}}\, dz = \int_{B(x,1)\setminus B(x,\varepsilon)} \frac{\delta(z)}{|x-z|^{n+1}}\, dz$$

In this annulus



$$\delta(z) \le |x-z|$$
$$|x-z| \ge \varepsilon$$

So
$$\frac{\delta(z)}{|x-z|^{n+1}} \le \frac{|x-z|}{|x-z|^{n+1}} = \frac{1}{|x-z|^n} \le \frac{1}{\varepsilon^n}$$
the function that we integrate is bounded. □

The next problem shows a typical and a very powerful application of the Fubini theorem.

Problem 8 Assume that $F \subset \mathbb{R}^n$ is closed and that $|F^c| = |\mathbb{R}^n \setminus F| < \infty$. Prove that

$$\int_{\mathbb{R}^n} \frac{\delta(z)}{|x-z|^{n+1}}\, dz < \infty$$

for almost all $x \in F$.

Proof This is a general idea that often works. If we want to prove that

$$\int_{\mathbb{R}^n} f(x,y)\, dx < \infty$$

for almost all $y \in F$, then it suffices to show that

$$\int_F \int_{\mathbb{R}^n} f(x,y)\, dx\, dy = \int_{\mathbb{R}^n} \int_F f(y,x)\, dy\, dx < \infty .$$

We will apply this to our situation.

First note that $\delta(z) = 0$ for $z \in F$ so

$$\int_{\mathbb{R}^n} \frac{\delta(z)}{|x-z|^{n+1}}\, dz = \int_{F^c} \frac{\delta(z)}{|x-z|^{n+1}}\, dz$$

and it suffices to show that

$$I = \int_F \int_{F^c} \frac{\delta(z)}{|x-z|^{n+1}}\, dz\, dx < \infty$$

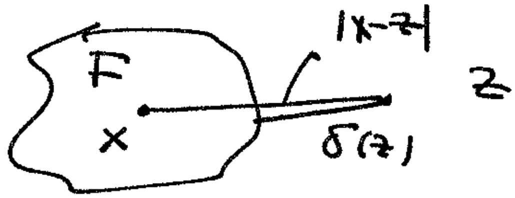
since it will readily imply that

$$\int_{F^c} \frac{\delta(z)}{|x-z|^{n+1}}\, dz < \infty \text{ for a.e. } x \in F.$$

According to the Fubini theorem we have

$$I = \int_{F^c} \int_F \frac{\delta(z)}{|x-z|^{n+1}}\, dx\, dz = \int_{F^c} \left(\int_F \frac{dx}{|x-z|^{n+1}} \right) \delta(z)\, dz.$$

If $z \in F^c$, then $|x-z| \geq \delta(z) > 0$



Therefore

$$\int_F \frac{dx}{|x-z|^{n+1}} \leq \int_{\{|x-z| \geq \delta(z)\}} \frac{dx}{|x-z|^{n+1}}$$

change of var. cf. Prob. 3

$$= \int_{\{|x| \geq \delta(z)\}} \frac{dx}{|x|^{n+1}} \leq \frac{C(n)}{\delta(z)}.$$

Problem 4

Therefore, $I \leq \int_{F^c} \frac{C(n)}{\delta(z)} \delta(z)\, dz = C(n) |F^c| < \infty.$