Logarithm

If $t>0$ is a real number, then the unique real number x such that $e^x = t$ will be denoted by $\operatorname{Log} t$ so $\operatorname{Log} t$ is the standard logarithm that we know from calculus.

Definition If $a \in \mathbb{C}\setminus\{0\}$, then any number $z \in \mathbb{C}$ such that $e^z = a$ is called a logarithm of a and is denoted by $\log a$.

Now we will find a formula for $\log a$.

If $z = x+iy$, then

$$e^x e^{iy} = e^z = e^a = |a| \underbrace{e^{i \arg a}}_{\cos(\arg a) + i\sin(\arg a)}$$

$$\underbrace{\cos(\arg a) + i\sin(\arg a)}_{\text{polar rep. of } a}$$

So

$$e^x = |a|, \quad e^{iy} = e^{i\arg a}$$

$$x = \operatorname{Log}|a|, \quad y = \arg a$$

Thus

$$z = \operatorname{Log}|a| + i \arg a$$

and hence

$$\log a = \operatorname{Log}|a| + i\arg a = \underbrace{\operatorname{Log}|a| + i\operatorname{Arg} a}_{\operatorname{Log} a} + 2k\pi i$$

$\operatorname{Log} a = \operatorname{Log}|a| + i\operatorname{Arg} a$ is called the principal value of the logarithm so

$$\log a = \operatorname{Log} a + 2k\pi i = \operatorname{Log}|a| + (\operatorname{Arg} a + 2k\pi)\, i$$

Example $\operatorname{Log} i = \operatorname{Log}|i| + i\operatorname{Arg} i = \frac{\pi}{2} i$

$\log i = (2k + \frac{1}{2})\pi i$, $k \in \mathbb{Z}$

Derivative of the logarithm

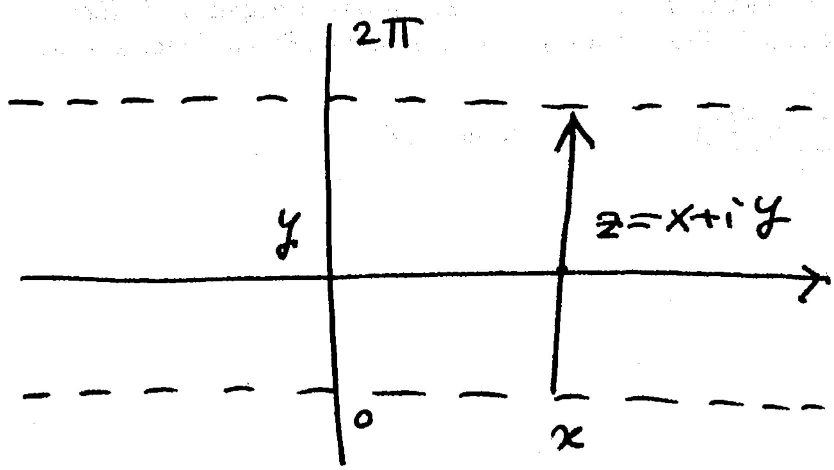
Let us investigate the function e^z in the strip

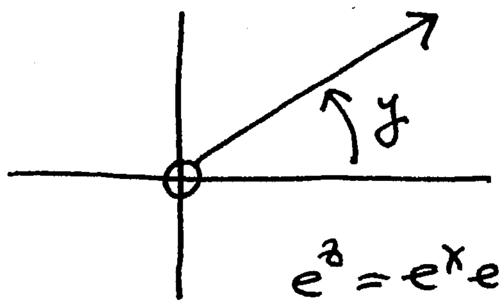
$\Delta = \{z \mid 0 < \operatorname{im} z < 2\pi\}$

Since $e^{z_1} = e^{z_2} \iff z_2 = z_1 + 2k\pi i$ it follows that e^z is one-to-one in Δ.

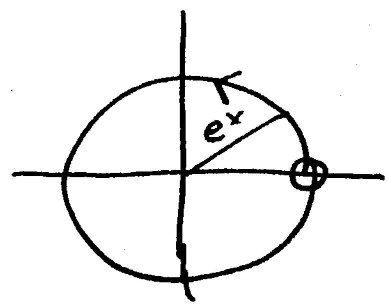
Also $(e^z)' = e^z \neq 0$ so the Jacobian is different than 0 (because for holomorphic functions $\det Df(z) = |f'(z)|^2$). Therefore e^z is a diffeomorphism in Δ.

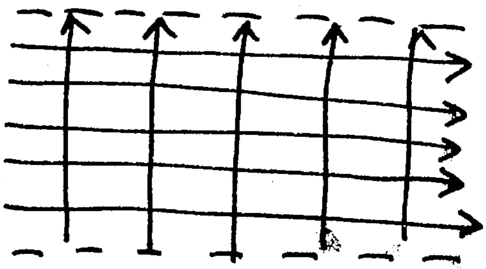
Let us check what is the image of Δ under e^z.

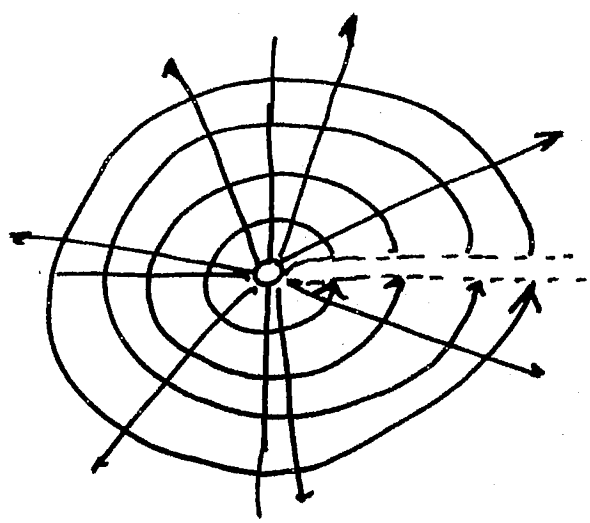




$e^z = e^x e^{iy}$







We proved that e^z maps as a diffeomorphism Δ onto $\mathbb{C} \setminus \mathbb{R}_+$

It follows from the definition of the principal value of the logarithm that

$$\mathrm{Log} : \mathbb{C} \setminus \mathbb{R}_+ \longrightarrow \Delta \qquad (*)$$

is the inverse diffeomorphism to

$$\Delta \xrightarrow{e^z} \mathbb{C} \setminus \mathbb{R}_+ .$$

Note that Log is complex differentiable.
(Indeed, if f is holomorphic with $f' \neq 0$, then f preserves angles and orientation so f^{-1} also preserves angles and orientation and hence f^{-1} is holomorphic.)
Therefore we can find the derivative of Log from the chain rule:

$$e^{\mathrm{Log}\, z} = z, \quad 1 = (e^{\mathrm{Log}\, z})' = \underbrace{e^{\mathrm{Log}\, z}}_{z} (\mathrm{Log}\, z)', \quad (\mathrm{Log}\, z)' = \frac{1}{z}.$$

We proved

<u>Theorem</u> The function $\mathrm{Log}\, z$ is holomorphic in $\mathbb{C} \setminus \mathbb{R}_+$. It is a diffeomorphism that maps $\mathbb{C} \setminus \mathbb{R}_+$ onto $\Delta = \{z \mid 0 < \operatorname{im} z < 2\pi\}$.
Moreover

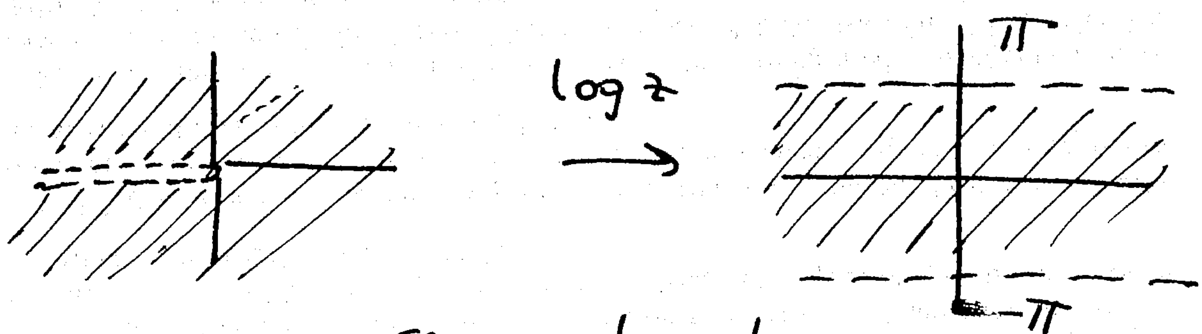
$$(\mathrm{Log}\, z)' = \frac{1}{z}$$

Now we will show how to expand logarithm in a power series.

By the same argument as above

$$\log z = \mathrm{Log}|z| + i \arg z, \quad -\pi < \arg z < \pi \qquad (*)$$

is a holomorphic diffeomorphism that maps $\mathbb{C} \setminus \mathbb{R}_-$ onto $\tilde{\Delta} = \{z \mid -\pi < \mathrm{Im} z < \pi\}$



$$(\log z)' = \frac{1}{z}.$$

Since $z \mapsto 1-z$ maps $D(0,1)$ onto $D(1,1) \subset \mathbb{C} \setminus \mathbb{R}_-$, the function $z \mapsto \log(1-z)$ defined with $(*)$ is holomorphic in $D(1,1)$ and

$$(\log(1-z))' = \frac{-1}{1-z} = -\sum_{n=0}^{\infty} z^n = \left(-\sum_{n=1}^{\infty} \frac{z^n}{n}\right)'$$

for $|z| < 1$. Therefore

$$\log(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n} + C$$

Taking $z=0$ yields and hence

$$\log(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}, \qquad |z|<1,$$

where $\log$ is defined by $(*)$.