Our aim is to discuss the notion of a holomorphic function $f:\Omega\to\overline{\mathbb{C}}$, where $\Omega\subset\overline{\mathbb{C}}$ is open. That includes the cases when $f(z_0)=\infty$, $f(\infty)=z_0$, $z_0\in\mathbb{C}$, $f(\infty)=\infty$. The idea is to represent $\overline{\mathbb{C}}$ as a smooth manifold (sphere) equipped with a complex structure. Since $\infty\in\overline{\mathbb{C}}$ will be a point on this manifold, it perfectly makes sense to talk about a smooth mapping defined in a neighborhood of ∞ and a smooth mapping that attains ∞ as a value.

While our discussion will be self-contained and we will not reffer to the notion of a manifold, those who are familiar with manifolds will see the connection.

We have two mappings that parametrize $\overline{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$.

$I:\mathbb{C}\to\overline{\mathbb{C}}\setminus\{\infty\}=\mathbb{C}$, $\quad I(z)=z$

$P:\mathbb{C}\to\overline{\mathbb{C}}\setminus\{0\}$, $P(z)=1/z$ with $P(0)=\infty$.

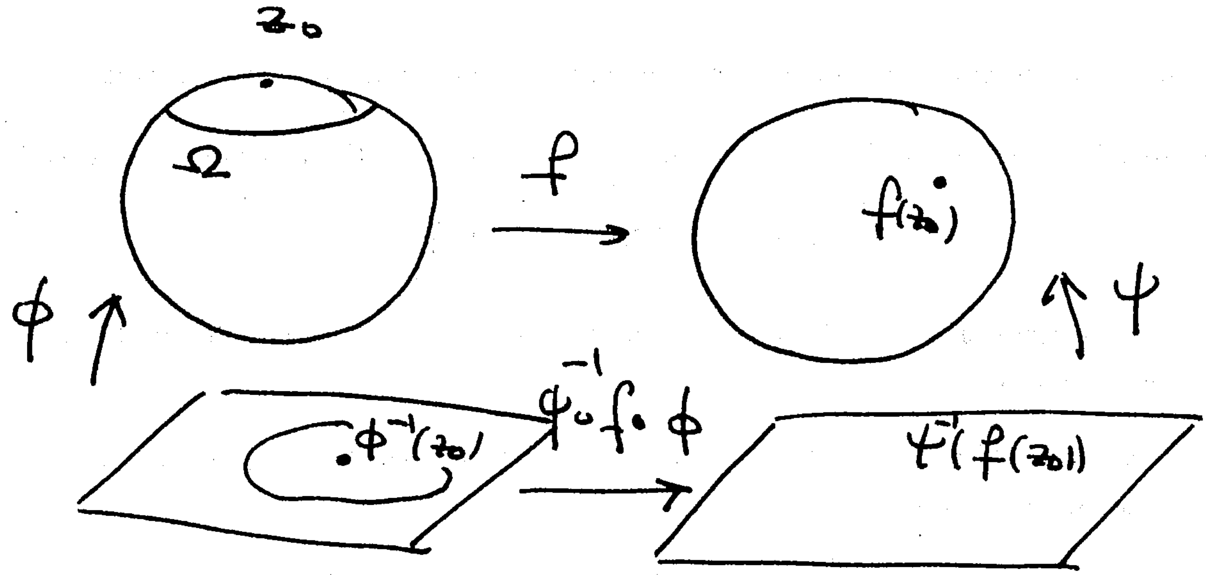
These mappings are homeomorphisms onto open subsets of $\overline{\mathbb{C}}$ due to the topology of $\overline{\mathbb{C}}$ that we discussed earlier.

Assume now that $f:\Omega\to\overline{\mathbb{C}}$ is a continuous map defined in an open neighborhood Ω of $z_0\in\overline{\mathbb{C}}$.

Let $\phi, \psi \in \{I, P\}$ be parametrizations of $\overline{\mathbb{C}}$ such that $z_0 \in \phi(\mathbb{C})$ and $f(z_0) \in \psi(\mathbb{C})$. Then

$$\psi^{-1} \circ f \circ \phi : \mathbb{C} \supset G \longrightarrow \mathbb{C}$$

is a complex valued function defined in a neighborhood G of $\phi^{-1}(z_0)$. It is easy to see that $G = \phi^{-1}(\Omega \cap \psi(\mathbb{C}))$.



<u>Definition</u> We say that f is holomorphic at $z_0 \in \overline{\mathbb{C}}$ is there are parametrizations ϕ, ψ as above such that $\psi^{-1} \circ f \circ \phi$ is complex differentiable at $\phi^{-1}(z_0)$.

While we define what it means to be holomorphic at $z_0 \in \overline{\mathbb{C}}$, the definition does not give the notion of the complex derivative at z_0.

Note that the parametrizations ϕ, ψ are not necessarily uniquely determined.

If $z_0 = 0$, we have to take $\phi(t) = I(t) = z$, and if $z_0 = \infty$, we have to take $\phi(t) = P(t) = 1/z$, so in these cases ϕ is uniquely determined. However, if $z_0 \in \mathbb{C} \setminus \{0\}$ we can take $\phi_1 = I$ or $\phi_2 = P$. Similar discussion applies to the uniqueness of ψ.

Note that whether f is holomorphic at z_0 does not depend on the choice of parametrizations ϕ and ψ, because

$$\psi_2^{-1} \circ f \circ \phi_2 = (\psi_2^{-1} \circ \psi_1) \circ \psi_1^{-1} \circ f \circ \phi_1 \circ (\phi_1^{-1} \circ \phi_2)$$

and the functions $\phi_1^{-1} \circ \phi_2$ and $\psi_2^{-1} \circ \psi_1$ are holomorphic.

Thus if $\psi_1^{-1} \circ f \circ \phi_1$ is holomorphic at $\phi_1^{-1}(z_0)$, then $\psi_2^{-1} \circ f \circ \phi_2$ is holomorphic at $\phi_2^{-1}(z_0)$ and similarly, if $\psi_2^{-1} \circ f \circ \phi_2$ is holomorphic at $\phi_2^{-1}(z_0)$, then $\psi_1^{-1} \circ f \circ \phi_1$ is holomorphic at $\phi_1^{-1}(z_0)$.

To see that the compositions $\phi_1^{-1} \circ \phi_2$ and $\psi_2^{-1} \circ \psi_2$ are holomorphic, note that they are one of the following maps

$$I^{-1}\circ I : \mathbb{C} \longrightarrow \mathbb{C}\,, \quad I^{-1}\circ I(z) = z$$
$$P^{-1}\circ I : \mathbb{C}\setminus\{0\} \longrightarrow \mathbb{C}, \quad P^{-1}\circ I(z) = 1/z$$
$$(*) \quad P^{-1}\circ P : \mathbb{C} \longrightarrow \mathbb{C}\,, \quad P^{-1}\circ P(z) = z$$
$$I^{-1}\circ P : \mathbb{C}\setminus\{0\} \longrightarrow \mathbb{C}, \quad I^{-1}\circ P(z) = 1/z.$$

Let us check for example $(*)$

$$P^{-1}\circ P(z) = \begin{cases} P^{-1}(1/z), & z\neq 0 \\ P^{-1}(\infty), & z=0 \end{cases} = \begin{cases} z\,, & z\neq 0 \\ 0\,, & z=0 \end{cases}$$

<u>Remark.</u> The mappings I, P define a smooth atlas on $\overline{\mathbb{C}}$ and hence a smooth structure on $\overline{\mathbb{C}}$. Since the change of variables $\psi^{-1}\circ\phi$ are not only smooth, but also holomorphic, the mappings I, P define a smooth complex structure on $\overline{\mathbb{C}}$.

If $z_0 \in \mathbb{C}$ and $f(z_0)\in\mathbb{C}$, we can take $\phi = I$ and $\psi = I$ so

$$\psi^{-1}\circ f\circ\phi = f$$

in a neighborhood of z_0 and hence f is holomorphic at z_0 in the sense of our new definition if and only if f is complex differentiable at z_0 in the classical sense.

While the above discussion is quite abstract, let
us see how we check practically whether a given map $f : \overline{\mathbb{C}} \supset \Omega \longrightarrow \overline{\mathbb{C}}$ is holomorphic.

Assume for simplicity that $f : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ is defined on the entire Riemann sphere and the question is:

| How do we check that $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is holomorphic at a given point $z_0 \in \overline{\mathbb{C}}$?

We have four different cases to consider

1. $\boxed{z_0 \neq \infty, \ f(z_0) \neq \infty}$ Then f is holomorphic at z_0 iff f is complex differentiable at z_0 in the classical sense.

2. $\boxed{z_0 = \infty, \ f(z_0) \neq \infty}$ Then f is holomorphic at $z_0 = \infty$ iff

$$g(z) = \begin{cases} f(1/z), & z \neq 0 \\ f(\infty), & z = 0 \end{cases}$$

is complex differentiable at $z = 0$.

(Indeed, $\psi^{-1} \circ f \circ \phi = I^{-1} \circ f \circ P = g$, $\phi^{-1}(z_0) = 0$)

3. $\boxed{z_0 \neq \infty, \ f(z_0) = \infty}$ Then f is holomorphic at z_0 iff

$$g(z) = \begin{cases} 1/f(z), & z \neq z_0 \\ 0, & z = z_0 \end{cases}$$

is complex differentiable at z_0.

$$\left(\psi^{-1} \circ f \circ \phi(z) = P^{-1} \circ f \circ I(z) = P^{-1}(f(z)) = \begin{cases} \frac{1}{f(z)}, & z \neq z_0 \\ 0, & z = z_0 \end{cases}, \right.$$

$$\left. \phi^{-1}(z_0) = z_0 \right).$$

④ $\boxed{z_0 = \infty,\ f(z_0) = \infty}$ Then f is holomorphic at $z_0 = \infty$ iff

$$g(z) = \begin{cases} 1/f(1/z), & z \neq 0 \\ 0, & z = 0 \end{cases}$$

is complex differentiable at $z = 0$.

$$\left(\psi^{-1} \circ f \circ \phi(z) = P^{-1} \circ f \circ P(z) = 1/f(1/z), \right.$$

$$\left. \phi^{-1}(z_0) = P^{-1}(\infty) = 0 \right).$$

Now we will discuss an important example.

<u>Definition</u> Functions of the form $W(z) = P(z)/Q(z)$, where P and Q are complex polynomials are called rational functions.

Rational function define in a natural way continuous mappings

$$W : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$$

Indeed, using the fundamental theorem of calculus we can write

$$W(z) = \frac{P(z)}{Q(z)} = A \, \frac{(z-z_1)\dots(z-z_k)}{(z-\omega_1)\dots(z-\omega_n)}$$

and we can assume that $A \neq 0$ and $z_i \neq \omega_j$. The function W is defined for $z \in \mathbb{C} \setminus \{\omega_1, \dots, \omega_n\}$.

However,

$$W(\omega_i) := \lim_{z \to \omega_i} W(z) = \infty$$

$$W(\infty) := \lim_{z \to \infty} W(z) = \begin{cases} A & \text{if } k = n \\ \infty & \text{if } k > n \\ 0 & \text{if } k < n \end{cases}$$

allows us to extend W as a continuous map

$$W : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}.$$

Theorem Any rational function $W : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is holomorphic on $\overline{\mathbb{C}}$

Remark Later we will prove that every holomorphic function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ must be a rational function.

Proof Clearly W is holomorphic at $z \in \mathbb{C} \setminus \{\omega_1, \dots, \omega_n\}$ and it remains to prove that W is holomorphic at $z = \omega_i$, $i = 1, 2, \dots, n$ and at $z = \infty$.

To prove differentiability at ω_i we need to prove differentiability of

$$\frac{1}{W(z)} = \frac{1}{A} \frac{(z-\omega_1)\dots(z-\omega_n)}{(z-z_1)\dots(z-z_k)}$$

at $z = \omega_i$ (case 3) which is obvious.

It remains to prove differentiability at ∞. To this end, we need to prove differentiability of

$$W\left(\tfrac{1}{z}\right) = A \frac{\left(\frac{1}{z}-z_1\right)\dots\left(\frac{1}{z}-z_k\right)}{\left(\frac{1}{z}-\omega_1\right)\dots\left(\frac{1}{z}-\omega_n\right)}$$

at 0 (Case 2 if $k \le n$, case 4 if $k > n$).

However, $z \longmapsto W(1/z)$ is a rational function and hence it is differentiable at any finite point by previous considerations. In particular, it is differentiable at $z = 0$.