Zeros of analytic functions

Theorem Suppose that $f \in H(\Omega)$ and Ω is connected. If f is not constant equal zero, then the set

$$Z(f) = \{a \in \Omega \mid f(a) = 0\}$$

consists of isolated points (i.e. for every $a \in Z(f)$, there is $\varepsilon > 0$ such that $D(a, \varepsilon) \cap Z(f) = \{a\}$). Moreover, for every $a \in Z(f)$, there is $m \in \{1, 2, 3, \dots\}$ such that

$$f(z) = (z-a)^m g(z), \qquad z \in \Omega,$$

where $g \in H(\Omega)$, $g(a) \neq 0$.

Proof Assume that $a_0 \in Z(f)$. As we know, f can be expanded as a power series in a neighborhood of a_0

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a_0)^n, \quad |z-a_0| < \mathrm{dist}(a_0, \partial\Omega).$$

Lemma There is m such that $c_m \neq 0$.

Proof Suppose to the contrary that $c_m = 0$ for all m. We will arrive to a contradiction by showing that $f = 0$ in Ω. Let $A \subset \Omega$ be the set of all points $a \in \Omega$ such that $f = 0$ in a neighborhood of a.

It suffices to prove that $A = \Omega$. Since $A \neq \emptyset$ (because $a_0 \in A$), and Ω is connected, it suffices to show that A is an open and a closed subset of Ω. A is open by its definition. To show that A is closed, it suffices to show that

$$(A \ni a_i \to a \in \Omega) \Longrightarrow a \in A \qquad (*)$$

If $a_i = a$ for some i, then $a \in A$ so we may assume that $a_i \neq a$ for all i. We have

$$f(z) = \sum_{n=0}^{\infty} b_n (z-a)^n, \quad |z-a| < \operatorname{dist}(a, \partial\Omega).$$

It suffices to show that $b_n = 0$ for all n as it will clearly imply that $a \in A$.

Suppose to the contrary that $b_m \neq 0$ for some m and assume that m is the least integer with this property (i.e. $m = 0$ or $b_0 = \ldots = b_{m-1} = 0$). We have

$$f(z) = (z-a)^m \underbrace{\sum_{n=0}^{\infty} b_{n+m} (z-a)^n}_{g(z)}$$

and hence

$$0 = f(a_i) = \underbrace{(a_i - a)^m}_{\neq 0} g(a_i), \quad g(a_i) = 0$$

$$0 = g(a_i) \longrightarrow g(a), \quad g(a) = 0$$

$$b_m = 0.$$

which is a contradiction. The proof of the lemma is complete. $\square$

According to the lemma, if $a_0 \in Z(f)$, then $c_m \neq 0$ for some m. If m is the least integer with this property, then

$$f(z) = (z - a_0)^m \underbrace{\sum_{n=0}^{\infty} c_{n+m} (z - a_0)^n}_{g(z)}, \quad |z - a_0| < dist(a_0, \partial\Omega)$$

$$g(z) = \begin{cases} \dfrac{f(z)}{(z - a_0)^m}, & z \neq a_0 \\ c_m, & z = a_0 \end{cases}$$

Note that $g(a_0) \neq 0$ and $g \in H(\Omega)$. This is obvious that $g \in H(\Omega \setminus \{a_0\})$, but g is also holomorphic near a_0 because it is defined there through a convergent power series so $g \in H(\Omega)$.

Finally, since $g(a_0) \neq 0$, there is $\varepsilon > 0$ such that $g \neq 0$ in $D(a_0, \varepsilon)$ and hence $f \neq 0$ in $D(a_0, \varepsilon) \setminus \{a_0\}$ proving that $D(a_0, \varepsilon) \cap Z(f) = \{a_0\}$.

The proof of the theorem is complete.

Corollary If $f, g \in H(\Omega)$ where Ω is connected and $f(z) = g(z)$ for z belonging to a set that has an accumulation point in Ω, then $f = g$ in Ω.

Indeed, $Z(f-g)$ has an accumulation point so we must have $f - g \equiv 0$ in Ω.

Corollary If $f, g \in H(\Omega)$, where Ω is connected and $f^{(n)}(z_0) = g^{(n)}(z_0)$ for some $z_0 \in \Omega$ and all $n = 0, 1, 2, \dots$, then $f = g$ in Ω

<u>Proof</u> In a neighborhood of z_0 we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z-z_0)^n = g(z)$$

Thus $f = g$ in Ω by the previous corollary. $\square$