Analysis III: homework # 1
Due day: Lecture, Friday September 9, 2016

NAME (print):

Circle the problems that you have solved:

1 2 3 4 5 6 7 8 9 10 11

The solutions must be written in a **legible** form. The front page **must** be returned. All the papers **must** be stapled. **If any of the conditions is not satisfied, the homework will burned and flushed away.** The homework **will not** be returned so you better have a copy.

Problem 1. Prove that in a normed linear space X, $\lim_{n\to\infty} \|x_n - x\| = 0$ implies $\lim_{n\to\infty} \|x_n\| = \|x\|$.

Problem 2. Let X be a real or complex normed space, $x, y \in X$ and $\lambda \in \mathbb{R}$. Find the limit

$$\lim_{n\to\infty} \Big(\|(n+\lambda)x + y\| - \|nx + y\| \Big).$$

Problem 3. Prove that the following two norms in the space $C[0,1]$ are not equivalent:

$$\|x\|_1 = \max_{t\in[0,1]} |x(t)| \qquad \text{and} \qquad \|x\|_2 = \int_0^1 |x(t)|\, dt.$$

Problem 4. Prove that ℓ^p, $1 \le p < \infty$ is separable.

Problem 5. Prove that ℓ^∞ is not separable.

Problem 6. Prove that the real spaces ℓ_3^1 and ℓ_3^∞ are not isometric.

Problem 7. Prove that the real spaces c_0 and c are not isometric.

Problem 8. Prove that if X is infinitely dimensional linear space, then there is a norm $\|\cdot\|$ in X such that $(X, \|\cdot\|)$ is not complete.

Problem 9. Let X, Y be normed spaces, $\dim X = \infty$ and $Y \neq \{0\}$. Prove that there is a linear mapping $L : X \to Y$ which is unbounded.

Exercise 10. For a function $f : [-1,1] \to \mathbb{R}$ we define $T(f) : [-1,1] \to \mathbb{R}$ by the formula

$$T(f)(t) = t^2 f(t), \qquad \text{for } t \in [-1,1].$$

Consider T as a linear operator between the spaces X and Y in the following two cases
(a) $X = C([-1,1])$ and $Y = L^1([-1,1])$.
(b) $X = L^2([-1,1])$ and $Y = L^1([-1,1])$.
Prove that T is continuous and find its norm.

Problem 11. We know that if X is a Banach space and $T \in B(X)$, $\|T\| < 1$, then $I - T \in B(X)$ is an isomorphism, where I denotes the identity transformation. Show an example that this is not true without the assumption that X be a Banach space.