Analysis III: homework # 5
Due day: Lecture, Friday October 7, 2016

NAME (print):

Circle the problems that you have solved:

37 38 39 40 41 42 43 44 45

The solutions must be written in a **legible** form. The front page **must** be returned. All the papers **must** be stapled. **If any of the conditions is not satisfied, the homework will burned and flushed away.** The homework **will not** be returned so you better have a copy.

Problem 37. Given x, y in a complex inner product space, prove that

$$\langle x, y\rangle = \frac{1}{2\pi}\int_0^{2\pi} \|x + e^{it}y\|^2 e^{it}\, dt.$$

Problem 38. Let $\{x_n\}$ be an unbounded sequence in a Hilbert space H. Prove that $\{\langle x_n, x\rangle\}$ is unbounded for some $x \in H$.

Problem 39. Let $E \subset H$ be a non-empty, closed and convex subset of a Hilbert space H. As we know, for $x \in H$ there is a unique $P(x) \in E$ such that

$$\|x - P(x)\| = \inf_{y\in E} \|x - y\|.$$

Prove that

$$\|P(x) - P(y)\| \leq \|x - y\| \quad \text{for all } x, y \in H.$$

Problem 40. It is easy to see that $d(x, y) = \sqrt{|x - y|}$ is a metric on $\mathbb{R}$. Prove that the metric space $(\mathbb{R}, d)$ can be isometrically embedded into a Hilbert space.

Problem 41. Let $(X, \|\cdot\|)$ be a normed space of finite dimension n. Prove that there is a basis $(x, \ldots, x_n)$ and functionals $x_1^*, \ldots, x_n^* \in X^*$ such that $\|x_i\| = 1$, $\|x_i^*\| = 1$ for $i = 1, 2, \ldots, n$ and

$$\langle x_i^*, x_j\rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

Problem 42. Without using the Hahn-Banach theorem prove the following result:

Let $X \subset H$ be a closed subspace of a Hilbert space H and let $\ell \in X^*$ be a bounded linear functional on X. Then there is a unique $L \in H^*$ such that $\|L\|_{H^*} = \|\ell\|_{X^*}$ and

$$\langle L, x\rangle = \langle \ell, x\rangle \quad \textit{for all } x \in X.$$

Problem 43. Let X be a normed space and let ℓ be a linear and discontinuous functional on X. Prove that $\ker \ell$ is a dense subset of X.

Problem 44. Show an example of $f \in L^1(\mathbb{R}^n) \setminus L^2(\mathbb{R}^n)$.

Problem 45. Prove that if $f \in C(\mathbb{R}^n) \setminus L^2(\mathbb{R}^n)$, then there is an orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ of $L^2(\mathbb{R}^n)$ consisting of $C_0^\infty(\mathbb{R}^n)$ functions such that

$$\int_{\mathbb{R}^n} f(x)\varphi_n(x)\,dx = 0 \quad \text{for all } n = 1, 2, 3, \ldots.$$

Hint: *Use f to define a discontinuous functional and use Problem 43.*