Analysis III: homework # 6
Due day: Lecture, Friday October 14, 2016

NAME (print):

Circle the problems that you have solved:

46 47 48 49 50 51 52 53 54 55 56 57

The solutions must be written in a **legible** form. The front page **must** be returned. All the papers **must** be stapled. **If any of the conditions is not satisfied, the homework will burned and flushed away.** The homework **will not** be returned so you better have a copy.

Problem 46. Prove that if X is a normed space such that $\dim X = \infty$, then there is a linear functional on X which is discontinuous.

Problem 47. It is well known that there are homeomorhpisms of $\mathbb{R}^n$ that map sets of measure zero onto sets of positive measure. Use the Radon-Nikodym-Lebesgue theorem to prove that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism, then there is a set of measure zero $E \subset \mathbb{R}^n$ such that if $F \subset \mathbb{R}^n \setminus E$ has measure zero, them $f(F)$ has measure zero.

Problem 48. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite. Suppose that F is $\mathcal{M} \times \mathcal{N}$ measurable. Prove the following generalization of the Minkowski inequality

$$\left(\int_Y \left(\int_X |F(x,y)| \, d\mu(x) \right)^p d\nu(y) \right)^{1/p} \leq \int_X \left(\int_Y |F(x,y)|^p \, d\nu(y) \right)^{1/p} d\mu(x)$$

necessarily using the duality between L^p and L^q.

Problem 49. Prove that for a two-linear functional $B : X \times Y \to \mathbb{K}$ on a product or normed spaces (i.e. functional that is linear with respect to each variable) the following conditions are equivalent.

(a) B is continuous.
(b) B is continuous at $(0,0)$.
(c) There is a constant $C > 0$ such that $|B(x,y)| \leq C\|x\| \, \|y\|$.

Problem 50. Let X be the vector space of all polynomials in one real variable, with norm

$$\|p\| = \int_0^1 |p(t)| \, dt.$$

Consider the two-linear functional $B : X \times X \to \mathbb{R}$ defined by

$$B(p,q) = \int_0^1 p(t)q(t) \, dt.$$

Show that B is continuous with respect to each variable p and q, but it is not continuous on $X \times X$.

Problem 51. **(a)** Let X be a normed space. Prove that if $P \in B(X)$ is a projection (i.e. $P^2 = P$), then the image $P(X)$ is a closed subspace of X.
(b) Show an example of a bounded operator in a Banach space $L \in B(X)$, whose image is not closed.

Problem 52. Let X be a normed space and $A \in B(X)$. Prove that the limit $\lim_{n\to\infty} \|A^n\|^{1/n}$ exists and equals $\inf_n \|A^n\|^{1/n}$

This ends material for Exam 1. Problems below and in the homeworks that will follow are for Exam 2.

Problem 53. Let X be an infinite dimensional Banach space and let $\{x_1, x_2, x_3, \dots\} \subset X$ be linearly independent vectors. Prove that the subspace $Y = \operatorname{span}\{x_1, x_2, x_3, \dots\}$ is not closed in X.

Problem 54. Let $1 \leq p < q < \infty$. Prove that $L^q([0,1])$ is a vector subspace of $L^p([0,1])$ of first category.

Exercise 55. Let X be a closed subspace in $L^1[0,1]$ such that for every $f \in X$, $f \in L^p[0,1]$ for some $p > 1$. Prove that $X \subset L^p[0,1]$ for some $p > 1$.

Problem 56. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at every point of $\mathbb{R}$. Prove that f is Lipschitz on some interval $[a,b] \subset \mathbb{R}$, $a < b$.

Problem 57. Let X be a Banach space and let $\Lambda : X \to \ell^\infty$ be a linear operator, so that $\Lambda(x) = (\Lambda_1(x), \Lambda_2(x), \dots)$ is a bounded sequence of real numbers for every $x \in X$. Prove that the operator Λ is bounded if and only if each linear functional Λ_n is bounded.