Analysis III: homework # 7
Due day: Lecture, Friday November 18, 2016

NAME (print):

Circle the problems that you have solved:

58 59 60 61 62 63 64 65 66 67 68

Problem 58. Prove that for $0 < p < 1$, $L^p([0,1])$ is a complete metric space with respect to the metric

$$d(f,g) = \int_0^1 |f(t) - g(t)|^p \, dt$$

Prove that there are no nonzero continuous functionals on $L^p([0,1])$ when $0 < p < 1$.

Problem 59. Let Y be a subspace of a real Banach space X and let $\Lambda : Y \to \mathbb{R}^n$ be a bounded linear operator. Show that Λ can be extended to a bounded linear operator $\tilde{\Lambda} : X \to \mathbb{R}^n$ such that $\|\tilde{\Lambda}\| \leq \sqrt{n}\|\Lambda\|$.

Problem 60. Prove that there is a functional $0 \neq x^* \in (L^\infty(\mathbb{R}))^*$ such that $\langle x^*, f \rangle = 0$ for every bounded continuous function.

Problem 61. Let f be a bounded linear functional on a subspace M of a Hilbert space H. Prove that f has a unique norm-preserving extension to a bounded linear functional on H, and that this extension vanishes on $M^\perp$.

Exercise 62. Prove that every bounded functional on c_0 has unique norm-preserving extension to a functional on ℓ^∞.

Exercise 63. Let $G \subset \ell^1$ be a subspace consisting of sequences $(x_n)_{n=1}^\infty$ such that $x_1 = x_3 = x_5 = \ldots = 0$. Prove that every nonzero functional on G as infinitely many norm-preserving extensions to ℓ^1.

Exercise 64. Prove that if $f : A \to \mathbb{R}$ is an L-Lipschitz function defined on a subset $A \subset X$ of a metric space, then there is an L-Lipschitz function $F : A \to \mathbb{R}$ such that $F(x) = f(x)$ for all $x \in A$.

Problem 65. Let X and Y be Banach spaces. Let $D \subset B(X,Y)$ consists of operators that are one-to-one and have closed image. Prove that D is an open set.

Problem 66. If $1 < p < \infty$, prove that the unit ball in $L^p(\mu)$ is *strictly convex*, i.e

$$\|f\|_p = \|g\|_p = 1,\ f \neq g,\ \Rightarrow\ \|(f+g)/2\|_p < 1.$$

Show that this fails in $L^1(\mu)$, $L^\infty(\mu)$ and $C(X)$.

Problem 67. Fix $1 \leq p \leq \infty$. Let $(f_n)_{n=1}^\infty$ be a sequence of functions in $L^p(\mathbb{R})$, converging weakly to a function $f \in L^p(\mathbb{R})$. Prove that

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx \quad \text{for all } a < b.$$

Problem 68. Prove that the sequence $f_n(x) = \frac{1}{n}\chi_{[0,n]}$ has no subsequence that is weakly convergent in $L^1(\mathbb{R})$.