

Analysis III: homework # 1

Due day: Canvas, September 15

NAME:

Problem 1. Prove that in a normed linear space X , $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$ implies $\lim_{n \rightarrow \infty} \|x_n\| = \|x\|$.

Problem 2. Let X be a real or complex normed space, $x, y \in X$ and $\lambda \in \mathbb{R}$. Find the limit

$$\lim_{n \rightarrow \infty} \left(\|(n + \lambda)x + y\| - \|nx + y\| \right).$$

Problem 3. Prove that the following two norms in the space $C[0, 1]$ are not equivalent:

$$\|x\|_1 = \max_{t \in [0, 1]} |x(t)| \quad \text{and} \quad \|x\|_2 = \int_0^1 |x(t)| dt.$$

Problem 4. Prove that ℓ^p , $1 \leq p < \infty$ is separable.

Problem 5. Prove that ℓ^∞ is not separable.

Problem 6. Prove that the real spaces ℓ_3^1 and ℓ_3^∞ are not isometric.

Problem 7. Prove that the real spaces c_0 and c are not isometric.

Problem 8. Prove that if X is infinitely dimensional linear space, then there is a norm $\|\cdot\|$ in X such that $(X, \|\cdot\|)$ is not complete.

Problem 9. Let X, Y be normed spaces, $\dim X = \infty$ and $Y \neq \{0\}$. Prove that there is a linear mapping $L : X \rightarrow Y$ which is unbounded.

Exercise 10. For a function $f : [-1, 1] \rightarrow \mathbb{R}$ we define $T(f) : [-1, 1] \rightarrow \mathbb{R}$ by the formula

$$T(f)(t) = t^2 f(t), \quad \text{for } t \in [-1, 1].$$

Consider T as a linear operator between the spaces X and Y in the following two cases

(a) $X = C([-1, 1])$ and $Y = L^1([-1, 1])$.

(b) $X = L^2([-1, 1])$ and $Y = L^1([-1, 1])$.

Prove that T is continuous and find its norm.

Problem 11. We know that if X is a Banach space and $T \in B(X)$, $\|T\| < 1$, then $I - T \in B(X)$ is an isomorphism, where I denotes the identity transformation. Show an example that this is not true without the assumption that X be a Banach space.