

Analysis III: homework # 3

Due day: October 17, Canvas

NAME (print):

Problem 20. Let X be a normed linear space. Prove that X is a Banach space if and only if the unit sphere $S = \{x \in X \mid \|x\| = 1\}$ equipped with the metric $d(x, y) = \|x - y\|$ is complete.

Problem 21. Let $x = (x_1, x_2, x_3, \dots) \in \ell^\infty$. Prove that the norm in the quotient space ℓ^∞/c_0 is given by

$$\|[x]\| = \limsup_{n \rightarrow \infty} |x_n|.$$

Problem 22. Let Y be a closed subspace of c consisting of constant sequences. Prove that the quotient space c/Y is isomorphic to c_0 .

Problem 23. Let $\{x_1, x_2, \dots, x_n\}$ be linearly independent elements of a normed linear space X . Prove that there is a constant $c > 0$ such that for every choice of scalars $\lambda_1, \dots, \lambda_n$, we have

$$\|\lambda_1 x_1 + \dots + \lambda_n x_n\| \geq c(|\lambda_1| + \dots + |\lambda_n|).$$

Problem 24. Let Y be a proper and closed subspace of a normed linear space X . Prove that

$$\sup_{0 \neq x \in X} \frac{\text{dist}(x, Y)}{\|x\|} = 1.$$

Problem 25. Let $M \subset C[0, 1]$ be a subset consisting of all functions f such that

$$\int_0^{1/2} f(t) dt - \int_{1/2}^1 f(t) dt = 1.$$

Prove that M is a closed and convex subset of $C[0, 1]$ that has no element of minimal norm.

Problem 26. Let M be the set of all $f \in L^1([0, 1])$ such that

$$\int_0^1 f(t) dt = 1.$$

Prove that M is convex, closed and it contains infinitely many elements of smallest norm.

Problem 27. Prove that for any integer $m > 0$ there is $\varphi \in C_0^\infty(B^n(0, 1))$ such that for any $\varepsilon > 0$ the function $\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(x/\varepsilon)$ satisfies

$$\varphi_\varepsilon * P = P \quad \text{for all polynomials } P \text{ of degree } \leq m.$$