

Analysis III: homework # 5

Due day: Canvas, November 14

NAME:

Problem 36. Let X be the vector space of all polynomials in one real variable, with norm

$$\|p\| = \int_0^1 |p(t)| dt.$$

Consider the two-linear functional $B : X \times X \rightarrow \mathbb{R}$ defined by

$$B(p, q) = \int_0^1 p(t)q(t) dt.$$

Show that B is continuous with respect to each variable p and q , but it is not continuous on $X \times X$.

Problem 37. Prove that for a two-linear functional $B : X \times Y \rightarrow \mathbb{K}$ on a product of normed spaces (i.e. functional that is linear with respect to each variable) the following conditions are equivalent.

- (a) B is continuous.
- (b) B is continuous at $(0, 0)$.
- (c) There is a constant $C > 0$ such that $|B(x, y)| \leq C\|x\| \|y\|$.

Problem 38. Let H be a real Hilbert space and let $A : H \rightarrow H$ be a bounded strictly positive operator, i.e. there is $\beta > 0$ such that

$$\langle Ax, x \rangle \geq \beta\|x\|^2 \quad \text{for all } x \in H.$$

Prove that A is an isomorphism and $\|A^{-1}\| \leq \beta^{-1}$.

Problem 39. Let H be a Hilbert space over $\mathbb{R}$ and let $B : H \times H \rightarrow \mathbb{R}$ be a two-linear functional (i.e. linear with respect to each variable) such that

$$|B(x, y)| \leq \alpha\|x\| \|y\| \quad \text{for some } \alpha > 0 \text{ and all } x, y \in H.$$

In addition assume that B is strictly positive, i.e.

$$B(x, x) \geq \beta\|x\|^2 \quad \text{for some } \beta > 0 \text{ and all } x \in H.$$

Prove that for every $y \in H$ there is a unique $u \in H$ such that

$$B(u, x) = \langle y, x \rangle \quad \text{for all } x \in H.$$

Moreover $\|u\| \leq \beta^{-1}\|y\|$.

Problem 40. Let $\{x_n\}$ be an unbounded sequence in a Hilbert space H . Prove that there is $x \in H$ such that the sequence $(\langle x_n, x \rangle)_{n=1}^\infty$ is unbounded.

Hint: Direct application of the Baire category theorem.

Problem 41. Let $E \subset H$ be a non-empty, closed and convex subset of a Hilbert space H . As we know, for $x \in H$ there is a unique $P(x) \in E$ such that

$$\|x - P(x)\| = \inf_{y \in E} \|x - y\|.$$

Prove that

$$\|P(x) - P(y)\| \leq \|x - y\| \quad \text{for all } x, y \in H.$$

Problem 42. It is easy to see that $d(x, y) = \sqrt{|x - y|}$ is a metric on $\mathbb{R}$. Prove that the metric space $(\mathbb{R}, d)$ can be isometrically embedded into a Hilbert space.

Hint: Find embedding into $L^2(\mathbb{R})$ using characteristic functions of intervals.

Problem 43. Let $(X, \|\cdot\|)$ be a normed space of finite dimension n . Prove that there is a basis $(x_1, \dots, x_n)$ and functionals $x_1^*, \dots, x_n^* \in X^*$ such that $\|x_i\| = 1$, $\|x_i^*\| = 1$ for $i = 1, 2, \dots, n$ and

$$\langle x_i^*, x_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

Hint: It is a difficult problem known as the Auerbach theorem.

Problem 44. Without using the Hahn-Banach theorem prove the following result:

Let $X \subset H$ be a closed subspace of a Hilbert space H and let $\ell \in X^*$ be a bounded linear functional on X . Then there is a unique $L \in H^*$ such that $\|L\|_{H^*} = \|\ell\|_{X^*}$ and

$$\langle L, x \rangle = \langle \ell, x \rangle \quad \text{for all } x \in X.$$