

Analysis III: homework # 6

Due day: Canvas, December 10

NAME:

Problem 45. Let X and Y be Banach spaces. Let $D \subset B(X, Y)$ consists of operators that are one-to-one and have closed image. Prove that D is an open set.

Problem 46. Let Y be a subspace of a real Banach space X and let $\Lambda : Y \rightarrow \mathbb{R}^n$ be a bounded linear operator. Show that Λ can be extended to a bounded linear operator $\tilde{\Lambda} : X \rightarrow \mathbb{R}^n$ such that $\|\tilde{\Lambda}\| \leq \sqrt{n}\|\Lambda\|$.

Problem 47. Prove that there is a functional $0 \neq x^* \in (L^\infty(\mathbb{R}))^*$ such that $x^*(f) = 0$ for every bounded continuous function.

Problem 48. Suppose that $L : L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)$, $1 \leq p, q < \infty$ is a non-zero bounded linear translation invariant operator i.e., $L(\tau_y f) = \tau_y(Lf)$ for all $f \in L^p$ and $y \in \mathbb{R}^n$, where $(\tau_y g)(x) = g(x + y)$. Prove that $q \geq p$.

Hint: Consider compactly supported function f and compute $\|f + \tau_y f\|_p$ when $|y|$ is large.

Problem 49. Let f be a bounded linear functional on a subspace M of a Hilbert space H . Prove that f has a unique norm-preserving extension to a bounded linear functional on H , and that this extension vanishes on $M^\perp$.

Exercise 50. Prove that every bounded functional on c_0 has unique norm-preserving extension to a functional on ℓ^∞ .

Exercise 51. Let $G \subset \ell^1$ be a subspace consisting of sequences $(x_n)_{n=1}^\infty$ such that $x_1 = x_3 = x_5 = \dots = 0$. Prove that every nonzero functional on G has infinitely many norm-preserving extensions to ℓ^1 .

Exercise 52. Prove that if $f : A \rightarrow \mathbb{R}$ is an L -Lipschitz function defined on a subset $A \subset X$ of a metric space, then there is an L -Lipschitz function $F : A \rightarrow \mathbb{R}$ such that $F(x) = f(x)$ for all $x \in A$.

Problem 53. We say that a Banach space is *strictly convex* if

$$\|x\| = \|y\| = 1, x \neq y \implies \|(x + y)/2\| < 1.$$

Show that the spaces $L^1([0, 1])$, $L^\infty([0, 1])$ and $C([0, 1])$ are not strictly convex (and hence they are not uniformly convex).

Problem 54. Fix $1 \leq p \leq \infty$. Let $(f_n)_{n=1}^\infty$ be a sequence of functions in $L^p(\mathbb{R})$, converging weakly to a function $f \in L^p(\mathbb{R})$. Prove that

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b f(x) dx \quad \text{for all } a < b.$$

Problem 55. Prove that the sequence $f_n(x) = \frac{1}{n} \chi_{[0, n]}$ has no subsequence that is weakly convergent in $L^1(\mathbb{R})$.