

Analysis III: homework # 7
Due day: Canvas, December 18, 2021

NAME:

Problem 56. Let H be a Hilbert space. Prove that if $x_n \rightharpoonup x$ in H , then there is a subsequence $\{x_{n_k}\}_k$, such that

$$\frac{x_{n_1} + x_{n_2} + \dots + x_{n_k}}{k} \rightarrow x.$$

Problem 57. Let H be a Hilbert space and let $\{x_n\}_n$ be a sequence of orthogonal vectors. Prove that the following conditions are equivalent:

- (a) $\sum_{n=1}^{\infty} x_n$ converges in the norm,
- (b) $\sum_{n=1}^{\infty} x_n$ converges weakly,
- (c) $\sum_{n=1}^{\infty} \|x_n\|^2$ converges.

Problem 58. Let X be a reflexive Banach space and $x^* \in X^*$. Prove that there is $x \in X$ such that $\|x\| = 1$ and $x^*(x) = \|x^*\|$.

Problem 59. Prove that the Cartesian product of any family of Hausdorff spaces is Hausdorff.

Problem 60. Prove that weak and weak-* topologies in Banach spaces are Hausdorff.

Problem 61. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of a Hilbert space H , and let X be a Banach space. Prove that if $A \in B(H, X)$ is such that $\sum_{n=1}^{\infty} \|Ae_n\|^2 < \infty$, then A is compact.

Problem 62. Let X, Y be Banach spaces, where X is reflexive. Prove that if $A \in B(X, Y)$ has the property that it maps weakly convergent sequences $\{x_n\}_n \subset X$ to convergent sequences $\{Ax_n\}_n \subset Y$, then A is compact.

Problem 63. Let $1 \leq p < \infty$. Let $(\lambda_1, \lambda_2, \lambda_3, \dots)$ be a bounded sequence of real numbers. Define the bounded linear operator $\Lambda : \ell^p \rightarrow \ell^p$ by

$$\Lambda(x_1, x_2, x_3, \dots) = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \dots).$$

Prove that Λ is compact if and only if $\lim_{k \rightarrow \infty} \lambda_k = 0$.