

Analysis 4: homework # 1
Due day: January 30, 11:59 pm

Solutions must be typed in LaTeX and submitted through Canvas.

Problem 1. Show an example of $f \in L^1(\mathbb{R}^n)$ such that $\mathcal{M}f$ is not integrable on $B(0, 1)$.

Problem 2. Prove that if μ is a finite positive Borel measure on $\mathbb{R}^n$, then

$$|\{x \in \mathbb{R}^n : \mathcal{M}\mu(x) > t\}| \leq \frac{5^n}{t} \mu(\mathbb{R}^n).$$

Problem 3. Prove that if $f \in L^p_{\text{loc}}(\mathbb{R}^n)$, $1 \leq p < \infty$, then the set of points that are not the p -Lebesgue points of f has measure zero.

Problem 4. Prove Lemma 2.18 which states that if $1 \leq p < \infty$, then $\|\tau_h f - f\|_p \rightarrow 0$ as $h \rightarrow 0$.

Problem 5. In the proof of the Lebesgue differentiation theorem we said that we could assume that $f \in L^1(\mathbb{R}^n)$. Show details how to conclude the Lebesgue differentiation theorem for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ if we know that the theorem is true for functions in $L^1(\mathbb{R}^n)$.

Problem 6. Provide details that Zygmund's inequality (Theorem 2.29) $\|\widetilde{\mathcal{M}}f\|_p \leq C\|f\|_p$, $1 < p < \infty$ implies that

$$\lim_{\substack{\text{diam } S \rightarrow 0 \\ S \in \mathcal{F}}} \int_S f(x-y) dy = f(x) \quad \text{a.e.}$$

Problem 7. Prove the Young's inequality: if $1 \leq p, q, r \leq \infty$, $q^{-1} = p^{-1} + r^{-1} - 1$, then

$$\|f * g\|_q \leq \|f\|_p \|g\|_r.$$

Definition. We define the *restricted maximal function* by

$$\mathcal{M}^\varepsilon f(x) = \sup_{r \geq \varepsilon} \int_{B(x,r)} |f(y)| dy.$$

Problem 8. Prove that if $f \in L^1(\mathbb{R}^n)$, then

$$|\mathcal{M}^\varepsilon f(x) - \mathcal{M}^\varepsilon f(y)| \leq \frac{n}{\varepsilon} |x - y| (\mathcal{M}^\varepsilon f(x) + \mathcal{M}^\varepsilon f(y)) \leq \frac{2n}{\omega_n \varepsilon^{n+1}} \|f\|_1 |x - y|.$$

This shows that the restricted maximal function of $f \in L^1$ is Lipschitz continuous.

Definition. Let $R > 0$, $\lambda \geq 0$. The *fractional maximal function* is defined as

$$\mathcal{M}_R^\lambda g(x) = \sup_{0 < r < R} r^\lambda \int_{B(x,r)} |g(z)| dz.$$

Problem 9. Prove that if $0 \leq \lambda < 1$ then there exists a constant $C = C(n, \lambda)$ such that

$$\int_B \frac{|g(z)|}{|x - z|^{n-1}} dz \leq C r^{1-\lambda} \mathcal{M}_{2r}^\lambda g(x) \quad \text{for all } g \in L^1(B) \text{ and all } x \in B,$$

where B is a ball in $\mathbb{R}^n$ of radius r . **Hint:** Mimic the proof of Lemma 1.17

Definition. For a set $E \subset \mathbb{R}^n$ and $s > 0$ we define the *Hausdorff content* as $H_\infty^s(A) = \inf \sum_i r_i^s$, where infimum is taken over all coverings of E by balls B_i with radii r_i .

Problem 10. Let $g \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, $R > 0$ and $\lambda \geq 0$. Prove that

$$H_\infty^{n-\lambda p}(\{M_R^\lambda g > t\}) \leq Ct^{-p} \int_{\mathbb{R}^n} g^p.$$

Hint: *Mimic the proof of weak type estimates for the maximal function.*

Problem 11. Prove that if $I_\alpha : L^p \rightarrow L^q$ is bounded, where $0 < \alpha < n$, $1 < p < n/\alpha$, then $q = np/(n - \alpha p)$. **Hint:** *Scale a function f by dilations and check how both sides of the inequality $\|I_\alpha f\|_q \leq C\|f\|_p$ behave.*