

Analysis 4: homework # 2

Due day: March 6, 11:59 pm

Solutions must be typed in LaTeX and submitted through Canvas.

Problem 12. Prove that if $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, then

$$Mf(x) \leq \liminf_{y \rightarrow x} Mf(y) \quad \text{for all } x \in \mathbb{R}^n,$$

where Mf is the Hardy-Littlewood maximal function.

Problem 13. Prove that if $f, g \in L^2(\mathbb{R}^n)$, then $f * g$ is uniformly continuous.

Problem 14. Prove that if $f \in L^1(\mathbb{R}^n)$ and $f * f = 0$, then $f = 0$ a.e. **Hint:** Use the Fourier transform.

Problem 15. Let $Pf = \sum_{|\alpha| \leq m} a_\alpha D^\alpha f$ be a partial differential operator with constant coefficients (at least one coefficient is different than 0). Prove that there are no functions $f \in \mathcal{S}_n$ satisfying $Pf = 0$ on $\mathbb{R}^n$. **Hint:** Use the Fourier transform.

Problem 16. Assume $f \in L^1 \cap L^\infty$ and $\hat{f} \geq 0$. Prove that $\hat{f} \in L^1$. **Hint:** Use the Gauss-Weierstrass summability method.

Problem 17. Evaluate $\int_0^\infty e^{-y^a} dy$, $a > 0$. Provide the answer in terms of the Γ function.

Problem 18. Find $\delta_a * \delta_b$, where δ_a and δ_b are Dirac measures concentrated at points $a, b \in \mathbb{R}^n$.

Problem 19. Prove that if μ_1, μ_2 are measures of finite total variation, then

$$\widehat{\mu_1 * \mu_2} = \widehat{\mu_1} \cdot \widehat{\mu_2},$$

Problem 20. Prove that if μ is a finite positive Borel measure on $\mathbb{R}^n$, then the function $h(x) = \widehat{\mu}(x)$ satisfies

$$\sum_{i,j=1}^n h(x_i - x_j) \xi_i \bar{\xi}_j \geq 0 \quad \text{for all } x_i \in \mathbb{R}^n \text{ and } \xi_i \in \mathbb{C}, i = 1, 2, \dots, n.$$