

Analysis 4: homework # 5 (take home exam)

Due day: May 2, 11:59:59.9999999 pm

Solutions must be typed in L^AT_EX and submitted through Canvas. There exam won't be accepted after the deadline, because I need time to grade it. You do not need to solve all problems, but if you want to get an A, you should solve 10 problems or more (with the emphasis on more). Include partial solutions and ideas. If you do not know how to complete a proof say that instead of providing unjustified conclusions. Such unjustified conclusions will count against you.

Problem 34. Prove that if $u \in \mathcal{S}'_n$ is homogeneous of degree a , then $D^\alpha u$ is homogeneous of degree $a - |\alpha|$.

Problem 35. Show directly that the distributions δ_0 and $\hat{\delta}_0$ are homogeneous and find their orders of homogeneity.

Problem 36. Prove that there is a continuous and nowhere differentiable function $f \in C_0(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, such that $\hat{f} \in C^\infty(\mathbb{R}^n)$.

Problem 37. Prove that if $\varphi \in C_0^\infty$ and

$$\int_{\mathbb{R}^n} x^\alpha \varphi(x) dx = 0 \quad \text{for all multiindices } \alpha,$$

then $\varphi = 0$.

Problem 38. Prove that there is $0 \neq \varphi \in \mathcal{S}_n$ such that

$$\int_{\mathbb{R}^n} x^\alpha \varphi(x) dx = 0 \quad \text{for all multiindices } \alpha$$

Hint: Consider φ with $0 \notin \text{supp } \hat{\varphi}$.

Problem 39. Prove the following result using the Bochner-Hecke theorem:

If P_k is a homogeneous harmonic polynomial of degree k , then for any $t > 0$ we have

$$\mathcal{F} \left(P_k(x) e^{-\pi t |x|^2} \right) (\xi) = t^{-k - \frac{n}{2}} (-i)^k P_k(\xi) e^{-\pi |\xi|^2 / t}.$$

Problem 40. Find the Fourier transform of the characteristic function of an interval $f = \chi_{[a,b]}$.

Problem 41. Use integration by parts to prove that

$$\left\| \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right\|_2 \leq \|\Delta \varphi\|_2 \quad \text{for } \varphi \in C_0^\infty(\mathbb{R}^n).$$

Problem 42. Prove that

$$\frac{d}{dx} (\log |x|) = \text{p.v.} \frac{1}{x} \quad \text{in } \mathcal{S}'_1.$$

Problem 43. Prove that

$$\left(\text{p.v.} \frac{1}{x} \right) [\varphi] = \lim_{R \rightarrow \infty} \int_{-R}^R \varphi'(x) \left| \log \frac{|x|}{R} \right| dx \quad \text{for } \varphi \in \mathcal{S}_1.$$

Problem 44. We proved that if $u \in \mathcal{S}'_n$ satisfies $\Delta u = f \in L^2(\mathbb{R}^n)$, then for any $1 \leq j, k \leq n$

$$\frac{\partial^2 u}{\partial x_j \partial x_k} = -R_j R_k f + P_{jk},$$

where P_{jk} is a polynomial. Prove that P_{jk} is a harmonic polynomial.

Problem 45. Represent the mixed Riesz transform $R_j R_k$ for $j \neq k$ as a convolution with a principal value distribution.

Problem 46. Let K be a function of one variable. Prove that for $n \geq 2$ we have

$$\int_{S^{n-1}} K(x \cdot \theta) d\sigma(\theta) = (n-1)\omega_{n-1} \int_{-1}^1 K(s|x|)(1-s^2)^{\frac{n-3}{2}} ds$$

for all $x \in \mathbb{R}^n \setminus \{0\}$.

Problem 47. Let $m \in \mathcal{M}_p(\mathbb{R}^n)$, $1 \leq p < \infty$, and let $\psi \in L^1(\mathbb{R}^n)$. Prove that $m * \psi \in \mathcal{M}_p(\mathbb{R}^n)$, and

$$\|m * \psi\|_{\mathcal{M}_p} \leq \|\psi\|_1 \|m\|_{\mathcal{M}_p}.$$

Problem 48. Fix $z \in \mathbb{C}$ with $\operatorname{re} z > 0$ and choose $z^{1/2}$ to have $\operatorname{re} z^{1/2} > 0$. Prove that the Fourier transform of

$$f(x) = z^{-1/2} e^{-\pi x^2/z}, \quad x \in \mathbb{R}$$

equals

$$\hat{f}(\xi) = e^{-\pi z \xi^2}, \quad \xi \in \mathbb{R}.$$

In particular for $\delta > 0$ we have

$$(0.1) \quad \left(\frac{e^{-\pi x^2/(1+i\delta)}}{(1+i\delta)^{1/2}} \right)^\wedge (\xi) = e^{-\pi(1+i\delta)\xi^2}.$$

Hint: Use Theorem 3.22 to prove that for $z = a > 0$, $\hat{f}(\xi) = e^{-\pi a \xi^2}$, and then use elementary properties of homomorphic functions.

Problem 49. Prove that if $p > 2$, then there is $f \in L^p(\mathbb{R})$ such that the distributional Fourier transform $\hat{f}$ is not given by a locally integrable function. **Hint:** Suppose that for every $f \in L^p$, $\hat{f}$ is a locally integrable function and use the closed graph theorem to prove that $\|\hat{f}\|_{L^1(B(0,1))} \leq C\|f\|_p$. Then obtain a contradiction using (0.1).