

Let us explain notation that will be used in this course.

①

In most of the cases functions will be defined on $\mathbb{R}^n$ and we will integrate with respect to the Lebesgue measure.

Unless specified otherwise, all functions will be complex valued. We need this assumption, because one of the main tools in harmonic analysis is the Fourier transform which when applied to a real valued function yields a complex valued one.

Balls and spheres will be denoted by $B^n(x, r)$ and $S^{n-1}(x, r)$ or just by $B(x, r)$, $S(x, r)$. We will use separate notation for the unit ball and the unit sphere in $\mathbb{R}^n$

$$B^n := B^n(0, 1), \quad S^{n-1} := S^{n-1}(0, 1).$$

The Lebesgue measure of a set $E \subset \mathbb{R}^n$ will be denoted by $|E|$. The volume of a unit ball in $\mathbb{R}^n$ will be denoted by

$$\omega_n := |B^n| \quad \text{so} \quad |B^n(0, r)| = \omega_n r^n$$

The surface measure will be denoted by σ , so $\sigma(S^{n-1}(0, r))$ is the $(n-1)$ -dimensional measure of $S^{n-1}(0, r)$.

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It is well known that

$$\sigma(S^{n-1}(0, r)) = n\omega_n r^{n-1} \quad (1)$$

Note that the expression on the right hand side equals the derivative of the volume of a ball of radius r , so (1) can be rewritten as

$$\sigma(S^{n-1}(0, r)) = \frac{d}{dr} |B^n(0, r)| \quad (2)$$

This formula has a nice geometric explanation.

The volume of the thin spherical shell

$\{r \leq |x| \leq r+h\}$ divided by its thickness h converges to $\sigma(S^{n-1}(0, r))$ as $h \rightarrow 0^+$. On the other hand, this limit equals

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{|\{r \leq |x| \leq r+h\}|}{h} &= \\ \lim_{h \rightarrow 0^+} \frac{|B^n(0, r+h)| - |B^n(0, r)|}{h} &= \frac{d}{dr} |B^n(0, r)|. \end{aligned}$$

Integration in the spherical coordinates is given by the formula

$$\begin{aligned} \int_{\mathbb{R}^n} f(x) dx &= \int_0^\infty \int_{S^{n-1}(0, r)} f(\theta) d\sigma(\theta) dr = \\ &= \int_0^\infty r^{n-1} \int_{S^{n-1}} f(r\theta) d\sigma(\theta) dr. \end{aligned}$$

The last equality is a consequence of the change of variables formula which

allows us to reduce the integration over a sphere of radius r to the integration over the unit sphere. ③

Integral average is denoted by the barred integral

$$\bar{\int}_E f d\mu := \frac{1}{\mu(E)} \int_E f d\mu.$$

Of course, we need to assume here that $0 < \mu(E) < \infty$. In particular

$$\bar{\int}_E f dx = \frac{1}{|E|} \int_E f dx$$

$$\bar{\int}_{S^{n-1}(0,r)} f d\sigma = \frac{1}{n\omega_n r^{n-1}} \int_{S^{n-1}(0,r)} f d\sigma.$$

χ_E will denote the characteristic function of a set E

δ_a is the Dirac measure centered at a i.e.,

$$\delta_a(E) = \begin{cases} 1 & \text{if } a \in E \\ 0 & \text{if } a \notin E \end{cases} \quad \text{so} \quad \int_{\mathbb{R}^n} f d\delta_a = f(a).$$

Open subsets of $\mathbb{R}^n$ will usually be denoted by Ω .

$C_0^\infty(\Omega)$ is the space of smooth functions with compact support in Ω . This space is often denoted in the literature by $C_c^\infty(\Omega)$. ④

$C_0(\mathbb{R}^n)$ is the space of continuous functions vanishing at infinity i.e., $f \in C_0(\mathbb{R}^n)$ if f is continuous on $\mathbb{R}^n$ and

$$\lim_{|x| \rightarrow \infty} f(x) = 0.$$

Unlike functions in $C_0^\infty(\mathbb{R}^n)$, functions in $C_0(\mathbb{R}^n)$ need not have compact support.

$C_0(\mathbb{R}^n)$ is a Banach space with respect to the supremum norm

$$\|f\|_\infty = \sup_{x \in \mathbb{R}^n} |f(x)|.$$

It is well known that $C_0^\infty(\mathbb{R}^n)$ is a dense subspace of $C_0(\mathbb{R}^n)$ i.e.,

$$\forall f \in C_0(\mathbb{R}^n) \forall \varepsilon > 0 \exists f_\varepsilon \in C_0^\infty(\mathbb{R}^n) \quad \|f - f_\varepsilon\|_\infty < \varepsilon.$$

We will use multiindex notation

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}, \quad x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$$

$$\alpha = (\alpha_1, \dots, \alpha_n), \quad |\alpha| = \alpha_1 + \dots + \alpha_n$$

Example $x = (x_1, x_2, x_3)$, $\alpha = (1, 0, 2)$

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$$D^\alpha u = \frac{\partial^3 u}{\partial x_1 \partial x_3^2}, \quad x^\alpha = x_1 x_3^2$$

With this notation the product rule takes a form of a binomial formula

$$D^\alpha (fg) = \sum_{\beta + \gamma = \alpha} \frac{\alpha!}{\beta! \gamma!} D^\beta f D^\gamma g$$

where

$$\alpha! = \alpha_1! \dots \alpha_n! \quad \beta + \gamma = (\beta_1 + \gamma_1, \dots, \beta_n + \gamma_n)$$

Increasing function = non-decreasing function
strictly increasing if $x < y \Rightarrow f(x) < f(y)$

Similarly, we define decreasing and strictly decreasing functions.

Sign of a real number x is

$$\text{sgn } x = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

By C we will denote a generic constant whose value may change in a single string of estimates.

By writing $C(n, m)$ we indicate that the constant depends on n and m only