

Maximal function

①

In the next few lectures we will study properties of the maximal function.

Definition For $f \in L^1_{loc}(\mathbb{R}^n)$, we define the Hardy-Littlewood maximal function by

$$Mf(x) = \sup_{r>0} \int_{B(x,r)} |f(y)| dy, \quad x \in \mathbb{R}^n$$

It is easy to prove that $Mf: \mathbb{R}^n \rightarrow [0, \infty]$ is measurable, although for some functions $f \in L^1_{loc}(\mathbb{R}^n)$ it may happen that $Mf(x) = +\infty$ for all $x \in \mathbb{R}^n$.

The operator $f \mapsto Mf$ is not linear, because $M(-f) = Mf$.

We say that an operator T is subadditive if

$$|T(f_1 + f_2)| \leq |Tf_1| + |Tf_2| \quad \text{a.e.}$$

$$|T(kf)| = |k| |Tf| \quad \text{a.e.}, \quad k \in \mathbb{C}$$

The maximal function is subadditive

$$M(f_1 + f_2) \leq Mf_1 + Mf_2$$

$$M(kf) = |k| Mf$$

There is no need to use absolute value here, because $Mf \geq 0$.

The next result plays a fundamental role ② in applications of the maximal function.

Theorem (Hardy-Littlewood)

If $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, then $Mf < \infty$ a.e.
Moreover

(a) If $f \in L^1(\mathbb{R}^n)$, then

$$|\{x \mid Mf(x) > t\}| \leq \frac{5^n}{t} \int_{\mathbb{R}^n} |f| \text{ for all } t > 0.$$

(b) If $f \in L^p(\mathbb{R}^n)$, $1 < p \leq \infty$, then $Mf \in L^p(\mathbb{R}^n)$ and

$$\|Mf\|_p \leq 2 \cdot 5^{n/p} \left(\frac{p}{p-1}\right)^{1/p} \|f\|_p \text{ when } 1 < p < \infty \quad (1)$$

$$\|Mf\|_\infty \leq \|f\|_\infty.$$

Remark 1 Note that the constant in (1) diverges to ∞ as $p \rightarrow 1^+$. In fact, if $f \in L^1(\mathbb{R}^n)$ is a non-zero function, then $Mf \notin L^1(\mathbb{R}^n)$, so the claim (b) does not extend to the case $p=1$ and in the case $p=1$ we only have a weaker estimate (a).

To prove that $Mf \notin L^1$, choose $R > 0$ such that

$$\lambda := \int_{B(0,R)} |f| > 0.$$

Then, for $|x| > R$, $B(0,R) \subset B(x, R+|x|)$ and hence

③

$$Mf(x) \geq \frac{\int_{B(x, R+|x|)} |f|}{|B(x, R+|x|)|} \geq \frac{1}{|B(0, R)|} \int_{B(0, R)} |f|$$

$$= \frac{1}{\omega_n (R+|x|)^n}$$

Since the function on the right hand side is not in $L^1(\mathbb{R}^n)$, we conclude that $Mf \notin L^1$.

Remark 2 If $g \in L^1(\mathbb{R}^n)$, then

$$t \chi_{\{|g(x)| > t\}} \leq |g|, \quad t > 0$$

so integration of this estimate yields the Chebyscher inequality

$$|\{x \mid |g(x)| > t\}| \leq \frac{1}{t} \int_{\mathbb{R}^n} |g|$$

Therefore, if T is a bounded operator in L^1 i.e., $\|Tf\|_1 \leq C \|f\|_1$ (we do not assume linearity of T) we obtain

$$|\{x \mid |Tf(x)| > t\}| \leq \frac{1}{t} \int_{\mathbb{R}^n} |Tf| \leq \frac{C}{t} \int_{\mathbb{R}^n} |f|$$

$$|\{x \mid |Tf(x)| > t\}| \leq \frac{C}{t} \int_{\mathbb{R}^n} |f|$$

Thus a bounded operator $T: L^1 \rightarrow L^1$ satisfies an estimate as in (a). Although M is not bounded in L^1 (because $Mf \notin L^1$)

when $0 \neq f \in L^1$), it still satisfies a weaker estimate (a). The estimate (a) is often called a weak type estimate (4)

In the proof of the Hardy-Littlewood theorem we will need two important results: the Cavalieri principle and the Br-Covering lemma.

Theorem (Cavalieri principle) If μ is a σ -finite measure on X and $\Phi: [0, \infty) \rightarrow [0, \infty)$ is increasing, absolutely continuous and $\Phi(0) = 0$, then

$$\int_X \Phi(|f|) d\mu = \int_0^\infty \Phi'(t) \mu(\{ |f| > t \}) dt$$

Remarks Recall that absolutely continuous functions are such that they are differentiable a.e., the derivative is integrable and

$$\int_s^t \Phi'(\tau) d\tau = \Phi(t) - \Phi(s)$$

for all s, t in the interval in which Φ is defined.

Proof The result follows immediately from the equality

(5)

$$\int_X \Phi(|f|) d\mu = \int_X \int_0^{|f(x)|} \Phi'(t) dt d\mu(x)$$

and the Fubini theorem. $\square$

Taking $\Phi(t) = t^p$ yields

Corollary If μ is a σ -finite measure on X and $0 < p < \infty$, then

$$\int_X |f|^p d\mu = p \int_0^\infty t^{p-1} \mu(\{|f| > t\}) dt$$

If B is a ball in a metric space, then by $5B$ we will denote a ball concentric with B and with 5 times the radius.

Theorem (5r-covering lemma)

Let $\mathcal{B}$ be a family of balls in a metric space such that

$$\sup \{ \text{diam } B \mid B \in \mathcal{B} \} < \infty$$

Then there is a subfamily $\mathcal{B}' \subset \mathcal{B}$ of pairwise disjoint balls such that

$$\bigcup_{B \in \mathcal{B}} B \subset \bigcup_{B \in \mathcal{B}'} 5B$$

⑥

Moreover, for every $B \in \mathcal{B}$, there is $B' \in \mathcal{B}'$ such that

$$B \cap B' \neq \emptyset \text{ and } B \subset 5B'$$

If the metric space is separable, then the family $\mathcal{B}'$ is countable (or finite) and we can arrange it as a (possibly finite) sequence

$$\mathcal{B}' = \{B_i : i=1, 2, \dots\} \text{ so}$$

$$\bigcup_{B \in \mathcal{B}} B \subset \bigcup_{i=1}^{\infty} 5B_i$$

Remark Here $\mathcal{B}$ can be a family of all open or all closed balls. In both cases the proof remains the same

Proof. Let

$$R := \sup \{ \text{diam } B \mid B \in \mathcal{B} \} < \infty$$

Divide the balls in the family $\mathcal{B}$ into pairwise disjoint subfamilies according to diameters of the balls

$$\mathcal{F}_j = \{ B \in \mathcal{B} \mid \frac{R}{2^j} < \text{diam } B \leq \frac{R}{2^{j-1}} \}$$

$$\text{Clearly, } \mathcal{B} = \bigcup_{j=1}^{\infty} \mathcal{F}_j$$

First, we select balls as large as possible, ⑦
so we choose balls from $\mathcal{F}_1$ and we take
as many as we can, so we define

$$\mathcal{B}_1 \subset \mathcal{F}_1$$

to be the maximal family of pairwise
disjoint balls. That is, balls in $\mathcal{B}_1$ are
pairwise disjoint and any ball in $\mathcal{F}_1$
intersects with a ball in $\mathcal{B}_1$.

Suppose we already selected families
 $\mathcal{B}_1, \dots, \mathcal{B}_{j-1}$.

In the construction of $\mathcal{B}_j$ we want to
select balls from $\mathcal{F}_j$, but we have to
make sure that the balls we select
do not intersect with any of the
previously selected balls. Thus we
select balls from

$$\hat{\mathcal{F}}_j = \{B \in \mathcal{F}_j \mid B \cap B' = \emptyset \text{ for all } B' \in \bigcup_{i=1}^{j-1} \mathcal{B}_i\}$$

Then we define $\mathcal{B}_j$ to be a maximal
family of pairwise disjoint balls
in $\hat{\mathcal{F}}_j$. Finally, we define

$$\mathcal{B}' = \bigcup_{j=1}^{\infty} \mathcal{B}_j.$$

Clearly, balls in the family $\mathcal{B}'$ are pairwise disjoint. ⑧

Let $B \in \mathcal{B}$. Then $B \in \mathcal{F}_j$ for some j . It follows from the definition of $\mathcal{B}_j$ that B intersects with a ball in $\bigcup_{i=1}^j \mathcal{B}_i$ (why?). If

$B \cap B' \neq \emptyset$, $B' \in \bigcup_{i=1}^j \mathcal{B}_i$, then

$$\text{diam } B \leq \frac{R}{2^{j-1}} = 2 \cdot \frac{R}{2^j} < 2 \text{ diam } B'$$

and hence $B \subset 5B'$. In particular

$$\bigcup_{B \in \mathcal{B}} B \subset \bigcup_{B' \in \mathcal{B}'} 5B'. \quad \square$$

Proof of the Hardy-Littlewood theorem

(*) Let $f \in L^1(\mathbb{R}^n)$ and let

$$E_t = \{x \mid Mf(x) > t\}.$$

For each $x \in E_t$, there is $r_x > 0$ such that

$$\int_{B(x, r_x)} |f| > t \quad \text{s.t.} \quad |B(x, r_x)| < \frac{1}{t} \int_{B(x, r_x)} |f|$$

Observe that $\sup_{x \in E_t} r_x < \infty$, because

$$f \in L^1(\mathbb{R}^n).$$

(9)

Clearly, $E_t \subset \bigcup_{x \in E_t} B(x, r_x)$

Applying the 5r-covering lemma to the family $\{B(x_i, r_{x_i})\}_{x_i \in E_t}$, we find pairwise disjoint balls $\{B(x_i, r_{x_i})\}_{i=1}^{\infty}$ such that

$$E_t \subset \bigcup_{x \in E_t} B(x, r_x) \subset \bigcup_{i=1}^{\infty} B(x_i, 5r_{x_i}),$$

and hence

$$\begin{aligned} |E_t| &\leq 5^n \sum_{i=1}^{\infty} |B(x_i, r_{x_i})| < \frac{5^n}{t} \sum_{i=1}^{\infty} \int_{B(x_i, r_{x_i})} |f| \\ &\leq \frac{5^n}{t} \int_{\mathbb{R}^n} |f|. \end{aligned}$$

This completes the proof of part (a).

Proof of (b) Let $f \in L^p(\mathbb{R}^n)$, $1 < p \leq \infty$.

Since obviously $\|Mf\|_{\infty} \leq \|f\|_{\infty}$,

we may assume that $1 < p < \infty$.

Fix $t > 0$. Then

$$|f| \leq |f_1| + \frac{t}{2}, \text{ where } f_1 = f \chi_{\{|f| > \frac{t}{2}\}}.$$

It is easy to check that $f_1 \in L^1(\mathbb{R}^n)$

so we can apply the estimate (a)

to f_1 . We have

$|f| \leq |f_1| + \frac{t}{2}$ so $Mf \leq Mf_1 + \frac{t}{2}$
and hence

$$\{Mf > t\} \subset \{Mf_1 > \frac{t}{2}\}.$$

Therefore,

$$\begin{aligned} |E_t| &= |\{Mf > t\}| \leq |\{Mf_1 > \frac{t}{2}\}| \stackrel{(a)}{\leq} \\ &\leq \frac{5^n}{t/2} \int_{\mathbb{R}^n} |f_1| = \frac{2 \cdot 5^n}{t} \int_{\{ |f| > \frac{t}{2} \}} |f|. \end{aligned}$$

Then, the Cavalieri principle yields

$$\begin{aligned} \int_{\mathbb{R}^n} |Mf|^p &= p \int_0^\infty t^{p-1} |\{Mf > t\}| dt \leq \\ &\leq p \int_0^\infty t^{p-1} \left(\frac{2 \cdot 5^n}{t} \int_{\{ |f| > t/2 \}} |f(x)| dx \right) dt = \\ &= 2 \cdot 5^n p \int_{\mathbb{R}^n} |f(x)| \int_0^{2|f(x)|} t^{p-2} dt dx = \\ &= 2^p \cdot 5^n \frac{p}{p-1} \int_{\mathbb{R}^n} |f(x)|^p dx. \end{aligned}$$

This completes the proof of part (b). $\square$

(11)

Remark In fact we proved the following estimate

$$|\{x \mid Mf(x) > t\}| \leq \frac{2 \cdot 5^n}{t} \int_{\{ |f| > \frac{t}{2} \}} |f|$$

which is somewhat stronger than the estimate in part (a).

Remark We proved the $5r$ -covering lemma for balls in a metric space. Is a version of this lemma true for cubes in $\mathbb{R}^n$?

In fact, cubes with sides parallel to coordinate axes in $\mathbb{R}^n$ are balls in the metric

$$d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|.$$

If we denote by $5Q$, a cube concentric with Q and with 5 times the diameter, the $5r$ -covering lemma applied to $(\mathbb{R}^n, d_\infty)$ yields

(12)

Corollary Let $\mathcal{F}$ be a family of open or closed cubes in $\mathbb{R}^n$ with sides parallel to coordinate axes and such that

$$\sup \{ \text{diam } Q : Q \in \mathcal{F} \} < \infty.$$

Then, there is a subfamily $\mathcal{F}' \subset \mathcal{F}$ of pairwise disjoint cubes such that

$$\bigcup_{Q \in \mathcal{F}} Q \subset \bigcup_{Q \in \mathcal{F}'} 5Q.$$

Moreover, for every $Q \in \mathcal{F}$, there is $Q' \in \mathcal{F}'$ such that $Q \cap Q' \neq \emptyset$ and $Q \subset 5Q'$.

Remark Although we are mainly interested in analysis in $\mathbb{R}^n$, it was convenient to prove the $5r$ -covering lemma in the generality of metric spaces, because this result gives immediately two different results in $\mathbb{R}^n$: a covering lemma for balls and a covering lemma for cubes.