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Fractional integration theorem

As a first application of the Hardy - Littlewood theorem we will prove a result about integration of Riesz potentials.

Definition For $0 < \alpha < n$, $n \geq 2$, we define the Riesz potential by

$$(I_\alpha f)(x) = \frac{1}{\delta(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad \text{where}$$

$$\delta(\alpha) = \frac{\pi^{\frac{n}{2}} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}$$

Remark At this moment, the particular value of the constant $\delta(\alpha)$ is not important to us. Just assume that this is some constant depending on n and α . The exact value of $\delta(\alpha)$ will be important much later. And do not worry, I will recall the definition and basic properties of the Γ function.

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Theorem (Hardy-Littlewood-Sobolev)

Let $\alpha > 0$, $1 < p < \infty$ and $\alpha p < n$.

Then there is a constant $C = C(n, p, \alpha)$ such that

$$\|I_\alpha f\|_{p^*} \leq C \|f\|_p \text{ for all } f \in L^p(\mathbb{R}^n),$$

where $p^* = \frac{np}{n-\alpha p}$.

Remark This result is also known as the Fractional Integration Theorem.

Exercise Prove that if $I_\alpha: L^p \rightarrow L^q$ is bounded, $1 < p < \infty$, $\alpha > 0$, $\alpha p < n$, then $q = p^*$.

Use the change of variables $x \mapsto tx$, $t > 0$ and check how both sides of the inequality

$$\|I_\alpha f\|_q \leq C \|f\|_p$$

change when you apply this change of variables. This is known as the scaling argument.

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Lemma If $0 < \alpha < n$ and $\delta > 0$, then, there is a constant $C = C(n, \alpha)$, such that

$$\int_{B(x, \delta)} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy \leq C \delta^\alpha M f(x), \quad x \in \mathbb{R}^n$$

Proof Fix $x \in \mathbb{R}^n$ and consider annuli

$$A(k) = B(x, \frac{\delta}{2^k}) \setminus B(x, \frac{\delta}{2^{k+1}}), \quad k=0, 1, \dots$$

We have

$$\int_{B(x, \delta)} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy = \sum_{k=0}^{\infty} \int_{A(k)} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy$$

$$\leq \sum_{k=0}^{\infty} \left(\frac{\delta}{2^{k+1}} \right)^{\alpha-n} \int_{A(k)} |f(y)| dy \leq$$

$$\leq \sum_{k=0}^{\infty} \left(\frac{\delta}{2^{k+1}} \right)^{\alpha-n} \int_{B(x, \delta/2^k)} |f(y)| dy$$

$$= \omega_n \sum_{k=0}^{\infty} \left(\frac{\delta}{2^{k+1}} \right)^{\alpha-n} \left(\frac{\delta}{2^k} \right)^n \int_{B(x, \delta/2^k)} |f(y)| dy$$

$$\leq \omega_n \delta^\alpha \left(\frac{1}{2} \right)^{\alpha-n} \left(\sum_{k=0}^{\infty} \frac{1}{2^{k\alpha}} \right) M f(x).$$

Proof of the theorem Fix $\delta > 0$. ④

Hölder's inequality and integration in spherical coordinates yields

$$\begin{aligned}
 & \int_{\mathbb{R}^n \setminus B(x, \delta)} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy \leq \\
 & \leq \|f\|_p \left(\int_{\mathbb{R}^n \setminus B(x, \delta)} \frac{dy}{|x-y|^{(n-\alpha)p'}} \right)^{1/p'} = \\
 & = \|f\|_p \left(n \omega_n \int_{\delta}^{\infty} s^{n-1} \frac{1}{s^{(n-\alpha)p'}} ds \right)^{1/p'} \\
 & = C(n, p, \alpha) \delta^{\alpha - (n/p)} \|f\|_p.
 \end{aligned}$$

We used here the fact that

$$n - (n-\alpha)p' < 0$$

which is equivalent to $\alpha p < n$.

The above estimate and the lemma yield

$$\begin{aligned}
 |I_{\alpha} f(x)| & \leq \int_{B(x, \delta)} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy + \int_{\mathbb{R}^n \setminus B(x, \delta)} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy \\
 & \leq C \left(\delta^{\alpha} M f(x) + \delta^{\alpha - (n/p)} \|f\|_p \right)
 \end{aligned}$$

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This estimate is true for any $\delta > 0$.
Take $\delta > 0$ such that

$$\delta^\alpha M f(x) = \delta^{\alpha - (n/p)} \|f\|_p$$

i.e.,

$$\delta = \left(\frac{M f(x)}{\|f\|_p} \right)^{-p/n}$$

and we obtain

$$|I_\alpha f(x)| \leq C (M f(x))^{1 - \frac{\alpha p}{n}} \|f\|_p^{\frac{\alpha p}{n}}$$

which is equivalent to

$$|I_\alpha f(x)|^{p^*} \leq C (M f(x))^p \|f\|_p^{\frac{\alpha p}{n} p^*}$$

Integrating both sides and using the Hardy-Littlewood theorem yields

$$\begin{aligned} \int_{\mathbb{R}^n} |I_\alpha f|^{p^*} &\leq C \|f\|_p^{\frac{\alpha p}{n} p^*} \int_{\mathbb{R}^n} |M f|^p \\ &\leq C \|f\|_p^{\frac{\alpha p}{n} p^*} \|f\|_p^p \end{aligned}$$

and it remains to observe that

$$\frac{\alpha p}{n} p^* + p = p^*.$$

□