

The Lebesgue differentiation theorem

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We will use now the maximal function to prove the Lebesgue differentiation theorem.

For a function f on $\mathbb{R}^n$ and $h \in \mathbb{R}^n$ we define a translation of f by τ as

$$\tau_h f(x) = f(x+h).$$

Lemma If $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, then
 $\|\tau_h f - f\|_p \rightarrow 0$ as $h \rightarrow 0$.

This result is obvious if $f \in C_0^\infty(\mathbb{R}^n)$ and the general case follows from the density of C_0^∞ in L^p . We leave details as an exercise. In fact we proved this result in Analysis I.

Lemma If $f \in L^1(\mathbb{R}^n)$ and

$$f_r(x) = \int_{B(x,r)} f(y) dy,$$

then $f_r \rightarrow f$ in $L^1(\mathbb{R}^n)$ as $r \rightarrow 0^+$.

Moreover, we can find a sequence $r_i \rightarrow 0^+$ such that $f_{r_i} \rightarrow f$ a.e.

Proof We have

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$$\int_{\mathbb{R}^n} |f_r(x) - f(x)| dx =$$

$$\int_{\mathbb{R}^n} \left| \int_{B(x,r)} f(y) dy - f(x) \right| dx =$$

$$\int_{\mathbb{R}^n} \left| \int_{B(x,r)} (f(y) - f(x)) dy \right| dx \leq$$

$$\int_{\mathbb{R}^n} \int_{B(x,r)} |f(y) - f(x)| dy dx =$$

$$\int_{\mathbb{R}^n} \int_{B(0,r)} |f(x+y) - f(x)| dy dx =$$

$$\int_{B(0,r)} \underbrace{\int_{\mathbb{R}^n} |f(x+y) - f(x)| dx}_{\|T_y f - f\|_1} dy \leq$$

$$\sup_{|y| \leq r} \|T_y f - f\|_1 \rightarrow 0 \text{ as } r \rightarrow 0$$

by previous lemma.

We proved that $f_r \xrightarrow{r \rightarrow 0} f$ in L^1 .

Hence there is a sequence $r_i \rightarrow 0$.

such that $f_{r_i} \rightarrow f$ a.e. $\square$

The Lebesgue differentiation theorem is ③
a much stronger result.

Theorem (Lebesgue differentiation theorem)

If $f \in L^1_{loc}(\mathbb{R}^n)$, then

$$\lim_{r \rightarrow 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} f(y) dy = f(x)$$

for almost all $x \in \mathbb{R}^n$.

Proof By considering real and imaginary part of f separately, we may assume that f is real valued. Since the result is local in its nature we may also assume that $f \in L^1(\mathbb{R}^n)$ (why?).

Let

$$f_r(x) = \frac{1}{|B(x, r)|} \int_{B(x, r)} f(y) dy,$$

and define

$$\Omega f(x) = \limsup_{r \rightarrow 0} f_r(x) - \liminf_{r \rightarrow 0} f_r(x)$$

It suffices to prove that $\Omega f = 0$ a.e.

Indeed, equality $\Omega f = 0$ means that $f_r \rightarrow g$ a.e. to some function g .

Since $f_r \rightarrow f$ a.e. by previous lemma

we conclude that $f = g$ a.e. and

hence $f_r \rightarrow f$ a.e.

Observe that $\Omega f \leq 2Mf$. Recall the estimate ④

$$|\{Mf > t\}| \leq \frac{Cn!}{t} \int_{\mathbb{R}^n} |f|$$

Hence, for any $\varepsilon > 0$ we have

$$|\{\Omega f > \varepsilon\}| \leq |\{Mf > \varepsilon/2\}| \leq \frac{C}{\varepsilon} \int_{\mathbb{R}^n} |f|$$

Let h be a continuous function such that $\|f - h\|_1 < \varepsilon^2$. Since h is continuous, $\Omega h = 0$ and hence the subadditivity of Ω (check it!) yields

$$\Omega f \leq \Omega(f - h) + \Omega h = \Omega(f - h),$$

$$\begin{aligned} |\{\Omega f > \varepsilon\}| &\stackrel{s.o.}{\leq} |\{\Omega(f - h) > \varepsilon\}| \leq \frac{C}{\varepsilon} \int_{\mathbb{R}^n} |f - h| \\ &\leq C\varepsilon. \end{aligned}$$

Since it is true for any $\varepsilon > 0$, we conclude that $\Omega f = 0$ a.e. $\square$

If $f \in L^1_{loc}(\mathbb{R}^n)$ is real valued we define a representative of f at every point $x \in \mathbb{R}^n$ by

$$f(x) := \lim_{r \rightarrow 0} \sup \int_{B(x,r)} f(y) dy \quad (1)$$

According to the Lebesgue differentiation theorem this function coincides with f a.e. and hence it is indeed

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a representative of f in the class of functions that are equal a.e.

If f is complex valued, we define both real and imaginary part of f by (1) at every point of $\mathbb{R}^n$.

Definition Let $f \in L^1_{loc}(\mathbb{R}^n)$. We say that $x \in \mathbb{R}^n$ is a Lebesgue point of f if

$$\lim_{r \rightarrow 0} \int_{B(x,r)} |f(y) - f(x)| dy = 0 \quad (2)$$

provided $f(x)$ is defined by (1) and it is finite (i.e. $f(x) \in \mathbb{C}$).

Theorem If $f \in L^1_{loc}(\mathbb{R}^n)$, then the set of points which are not Lebesgue points of f has measure zero.

Proof It is easy to see that we can assume that f is real valued.

For a rational number $c \in \mathbb{Q}$ let E_c be the set of points $x \in \mathbb{R}^n$ such that the equality

$$\lim_{r \rightarrow 0} \int_{B(x,r)} |f(y) - c| dy = |f(x) - c| \quad (3) \quad \textcircled{6}$$

does not hold. According to the Lebesgue differentiation theorem $|E_c| = 0$ and hence also the set $E = \bigcup_{c \in \mathbb{Q}} E_c$ has measure zero.

It remains to prove that for any $x \in \mathbb{R}^n \setminus E$, (2) is satisfied.

This easily follows from (3) because we may approximate $f(x)$ by rational numbers. Indeed, if $x \in \mathbb{R}^n \setminus E$, then (3) is true for all $c \in \mathbb{Q}$.

Given $\varepsilon > 0$, let $c \in \mathbb{Q}$ be such that $|f(x) - c| < \varepsilon/3$. Then there is $\delta > 0$ such that for all $0 < r < \delta$ we have

$$\int_{B(x,r)} |f(y) - f(x)| dy \leq$$

$$\int_{B(x,r)} |f(y) - c| dy + |f(x) - c| \leq$$

$$(|f(x) - c| + \varepsilon/3) + |f(x) - c| < \varepsilon$$

The proof is complete. $\square$

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Definition We say that $x \in \mathbb{R}^n$ is a p -Lebesgue point of $f \in L^p_{loc}, 1 \leq p < \infty$, if

$$\lim_{r \rightarrow 0} \int_{B(x,r)} |f(y) - f(x)|^p dy = 0$$

Using a similar method as above (exercise) we obtain

Theorem If $f \in L^p_{loc}(\mathbb{R}^n), 1 \leq p < \infty$, then the set of points that are not p -Lebesgue points of f has measure zero.

Definition Let $E \subset \mathbb{R}^n$ be a measurable set. We say that $x \in \mathbb{R}^n$ is a density point of E if

$$\lim_{r \rightarrow 0} \frac{|B(x,r) \cap E|}{|B(x,r)|} = 1.$$

Applying the Lebesgue differentiation theorem to $f = \chi_E$ we obtain

Theorem Almost every point of a measurable set $E \subset \mathbb{R}^n$ is its density point and a.e. point of $\mathbb{R}^n \setminus E$ is not a density point of E .

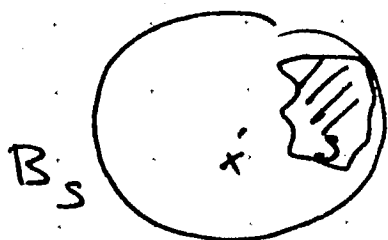
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The Lebesgue differentiation theorem shows that if $f \in L^1_{loc}(\mathbb{R}^n)$ then for almost every $x \in \mathbb{R}^n$, averages over balls centered at x converge to $f(x)$.

It is natural to ask whether balls can be replaced by other sets like cubes or even balls, ~~that~~ but not centered at x .

Definition We say that a family $\mathcal{F}$ of measurable subsets of $\mathbb{R}^n$ is regular at $x \in \mathbb{R}^n$ if

- (a) The sets in $\mathcal{F}$ are bounded and have positive measure
- (b) There is a sequence $\{S_i\} \subset \mathcal{F}$ with $|S_i| \rightarrow 0$
- (c) There is a constant $C > 0$ such that for every $S \in \mathcal{F}$ we have $|S| > C |B_S|$, where B_S is the smallest closed ball centered at x that contains S .



S occupies a substantial part of B_S .

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Note that if $|S_i| \rightarrow 0$, then the radius of B_{S_i} converges to 0 so the sets S_i approach to x .

Examples of families regular at x :

- Family of all balls centered at x
- Family of all balls that contain x
- Family of all cubes that contain x

Theorem If $f \in L^1_{loc}(\mathbb{R}^n)$, x is a Lebesgue point of f and $\mathcal{F}$ is a family regular at x , then

$$\lim_{\substack{S \in \mathcal{F} \\ |S| \rightarrow 0}} \int_S f(y) dy = f(x).$$

Proof We have

$$\begin{aligned} \left| \int_S f(y) dy - f(x) \right| &\leq \int_S |f(y) - f(x)| dy \\ &\leq \frac{1}{|S|} \int_{B_S} |f(y) - f(x)| dy \leq \\ &\leq \frac{|B_S|}{|S|} \int_{B_S} |f(y) - f(x)| dy \leq \\ &\frac{1}{C} \int_{B_S} |f(y) - f(x)| dy \xrightarrow{|S| \rightarrow 0} 0 \end{aligned}$$