

Zygmund's and Saks' Theorems

①

Recall that a family $\mathcal{F}$ of measurable subsets of $\mathbb{R}^n$ is regular at $x \in \mathbb{R}^n$ if

- (a) All sets in the family are bounded and have positive measure.
- (b) There is a sequence $\{S_i\}_i \subset \mathcal{F}$, $|S_i| \rightarrow 0$
- (c) There is a constant $C > 0$ such that $|S_i| \geq C |B_{S_i}|$, where B_{S_i} is the smallest closed ball centered at x that contains $S_i \in \mathcal{F}$.

Note that because of (c), $|S_i| \rightarrow 0$ if and only if $\text{diam } S_i \rightarrow 0$. Moreover, the sets S_i converge to x .

We proved that if a family $\mathcal{F}$ is regular at $x \in \mathbb{R}^n$, $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and x is a Lebesgue point of f , then

$$\lim_{\substack{S \in \mathcal{F} \\ \text{diam } S \rightarrow 0}} \int_S f(y) dy = f(x) \quad (1).$$

With a family $\mathcal{F}$ is regular at 0, then we can associate a maximal function

$$M_{\mathcal{F}} f(x) = \sup_{S \in \mathcal{F}} \int_S |f(x-y)| dy = \sup_{S \in \mathcal{F}} \int_{x-S} |f(y)| dy \quad (2)$$

That is, at any point $x \in \mathbb{R}^n$ we take the supremum of averages over the family $\{x-S\}_{S \in \mathcal{F}}$ which is regular at x .

It is easy to prove that

$$M_{\mathcal{F}} f \leq C M f \quad (2)$$

and hence $M_{\mathcal{F}}$ is bounded in L^p , $1 < p \leq \infty$ and it satisfies a weak type estimate for $p=1$. The proof of (2) follows from a routine argument

$$\begin{aligned} \int_S |f(x-y)| dy &= \frac{1}{|S|} \int_{B_S} |f(x-y)| dy = \\ &= \frac{|B_S|}{|S|} \int_{B_S} |f(x-y)| dy \leq \frac{1}{C} \int_{B_S} |f(x-y)| dy \\ &= (B_S = B(0, r_S)) = \frac{1}{C} \int_{B(0, r_S)} |f(x-y)| dy \\ &= \frac{1}{C} \int_{B(x, r_S)} |f(y)| dy \leq \frac{1}{C} M f(x), \end{aligned}$$

and taking the supremum over all $S \in \mathcal{F}$ yields

$$M_{\mathcal{F}} f \leq \frac{1}{C} M f \quad (3)$$

(recall the convention that C may change its value in a single string of estimates so it does not matter

that we have C in (2) and $1/C$ in (3).

(3)

Consider now the family $\mathcal{F}$ of all rectangular boxes in $\mathbb{R}^n$ origin and have sides parallel to coordinate axes, and define the maximal function associated with $\mathcal{F}$

$$\tilde{M}f = \sup_{S \in \mathcal{F}} \int_S |f(x-y)| dy$$

Note that when $n > 1$, the family $\mathcal{F}$ is not regular at 0, because a rectangular box $S \in \mathcal{F}$ can be thin and long making the ratio $|B_S|/|S|$ arbitrarily large. In fact, the estimate

$$\tilde{M}f \leq C Mf$$

is not true with any constant, when $n > 1$. However, we have

Theorem (Zygmund) For $1 < p \leq \infty$, there is a constant $C = C(n, p) > 0$ such that

$$\|\tilde{M}f\|_p \leq C \|f\|_p \quad (4)$$

Moreover, if $f \in L^p_{loc}$, $1 < p \leq \infty$, then

$$\lim_{\substack{\text{diam } S \rightarrow 0 \\ S \in \mathcal{F}}} \int_S f(x-y) dy = f(x) \quad (5)$$

for almost all $x \in \mathbb{R}^n$.

First, we will prove how to conclude (5) from (4). Note that since the family $\mathcal{F}$ is not regular at 0, we cannot conclude (5) from (1).

By considering the real and the imaginary part separately, we may assume that f is real valued. We have

$$0 \leq \limsup_{\substack{\text{diam } S \rightarrow 0 \\ S \in \mathcal{F}}} \int_S f(x-y) dy - \liminf_{\substack{\text{diam } S \rightarrow 0 \\ S \in \mathcal{F}}} \int_S f(x-y) dy \leq 2\tilde{M}f(x),$$

and hence (5) follows from (4) by a similar argument to the one we used in the proof of the Lebesgue differentiation theorem via a maximal function estimate. We leave details as an exercise.

Therefore, it remains to prove (4). In dimension $n=1$ we have

$$\tilde{M}f(x) = \sup_{a, b > 0} \int_{x-a}^{x+b} |f(y)| dy.$$

This is so called non-centered maximal function. Since in dimension $n=1$, the family of intervals that contain x is regular at x , it follows that

$$\tilde{M}f \leq C Mf \quad (\text{only when } n=1),$$

and hence (4) is a consequence of the L^p estimate for Mf when $n=1$.

(5)

In the higher dimensional case, for the simplicity of notation we will prove (4) for $n=2$ only, but a similar argument works in any dimension. We have

$$\begin{aligned} \tilde{M}f(x_1, x_2) &= \sup_{\substack{a_1, b_1 > 0 \\ a_2, b_2 > 0}} \int_{x_2-a_2}^{x_2+b_2} \int_{x_1-a_1}^{x_1+b_1} |f(y_1, y_2)| dy_1 dy_2 \\ &\leq \sup_{a_2, b_2 > 0} \int_{x_2-a_2}^{x_2+b_2} \left(\sup_{a_1, b_1 > 0} \int_{x_1-a_1}^{x_1+b_1} |f(y_1, y_2)| dy_1 \right) dy_2 \\ &= \tilde{M}_2 \tilde{M}_1 f(x_1, x_2), \end{aligned} \quad (6)$$

where on the right hand side we have an iteration of one-dimensional non-centered maximal functions. First, we fix variable y_2 , and take the maximal function with respect to y_1 and evaluate it at x_1 . Then, we take the maximal function with respect to y_2 and evaluate it at x_2 .

Now, inequality (4) follows from the corresponding inequality in dimension $n=1$ and the Fubini theorem.

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |\tilde{M}f(x_1, x_2)|^p dx_1 dx_2 \leq \quad (\text{Fubini and (6)}) \quad (6)$$

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |\tilde{M}_2 \tilde{M}_1 f(x_1, x_2)|^p dx_2 dx_1 \leq$$

(L^p estimate for $\tilde{M}_2$ with respect to variable x_2)

$$C \int_{\mathbb{R}} \int_{\mathbb{R}} |\tilde{M}_1 f(x_1, x_2)|^p dx_2 dx_1 = (\text{Fubini}) =$$

$$C \int_{\mathbb{R}} \int_{\mathbb{R}} |\tilde{M}_1 f(x_1, x_2)|^p dx_1 dx_2 \leq$$

(L^p estimate for $\tilde{M}_1$ with respect to variable x_1)

$$C \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x_1, x_2)|^p dx_1 dx_2$$

□

The above proof should be quite confusing, so please take a moment to make sure that you understand it.

Surprisingly, (5) is not true for $f \in L^1$ (when $n > 1$), and hence $\hat{M}f$ does not satisfy the weak type estimate for $p=1$. (Such an estimate would imply (5) as we demonstrated in the proof of the Lebesgue differentiation theorem).

(7)

Since $\tilde{M}f$ does not satisfy the weak type estimate in L^1 , $\tilde{M}$ is not bounded in L^1 . In fact, we have the following very surprising result.

Theorem (Saks) Let $\mathcal{F}$ be the family of rectangular boxes in $\mathbb{R}^n, n \geq 1$, that contain the origin and have sides parallel to coordinate axes. Then, the set of functions $f \in L^1(\mathbb{R}^n)$ such that

$$\limsup_{\substack{\text{diam } S \rightarrow 0 \\ S \in \mathcal{F}}} \int_S f(x-y)dy = +\infty \text{ for all } x \in \mathbb{R}^n,$$

is a dense G_δ subset of $L^1(\mathbb{R}^n)$. In particular, it is not empty.

This is a difficult result and we will not prove it.