

①

Theorem (Calderón-Zygmund decomposition)

Suppose that $f \in L^1(\mathbb{R}^n)$, $f \geq 0$ and $\alpha > 0$. Then, there is an open set Ω , and a closed set F , such that

$$(a) \mathbb{R}^n = \Omega \cup F, \quad \Omega \cap F = \emptyset$$

$$(b) f \leq \alpha \text{ a.e. in } F$$

(c) Ω is the union of pairwise disjoint open cubes $\Omega = \bigcup_{k=1}^{\infty} Q_k$, such that

$$\alpha < \int_{Q_k} f(x) dx \leq 2^n \alpha, \quad k=1, 2, 3, \dots$$

The result seems strange and there is no apparent reason why it should be useful, but it will play an important role in many proofs. Roughly speaking, the theorem says that for any $0 \leq f \in L^1$, $\mathbb{R}^n$ can be decomposed into a set F , where f is small, $f \leq \alpha$, and cubes Q_k , where integral averages of f are almost constant.

$$\alpha < \int_{Q_k} f \leq 2^n \alpha$$

This is helpful for example in estimating certain integral operators.

Decompose $\mathbb{R}^n$ into a grid of identical cubes, large enough to have (2)

$\int_Q f \leq \alpha$ for each cube Q in the grid.

Take any cube Q from this family and divide it into 2^n identical cubes. Let Q' be one of the cubes from the partition of Q . We have two possible cases

(1) $\int_{Q'} f > \alpha$ or (2) $\int_{Q'} f \leq \alpha$.

In the case (1), we include the open cube Q' in the family $\{Q_k\}_k$. Observe that

$$\alpha < \int_{Q'} f = \frac{1}{|Q'|} \int_{Q'} f = \frac{2^n}{|Q|} \int_{Q'} f \leq 2^n \int_Q f \leq 2^n \alpha$$

so the condition (c) is satisfied.

In the case (2) i.e., $\int_{Q'} f \leq \alpha$, we divide Q' into 2^n identical cubes and proceed as above. We continue this process infinitely many times or until it is terminated and we apply it to any cube Q in the grid and any cube in the partitions.

(3)

Let $\Omega = \bigcup_{k=1}^{\infty} Q_k$. Clearly Ω is open and the cubes Q_k satisfy property (c). It remains to prove that $f \leq \alpha$ a.e. in $\mathbb{R}^n \setminus \Omega$.

The set $F = \mathbb{R}^n \setminus \Omega$ consists of faces of the cubes (this set has measure zero) and points x such that there is a sequence of cubes $\tilde{Q}_i$ with the property that $x \in \tilde{Q}_i$, $\text{diam } \tilde{Q}_i \rightarrow 0$, $f_{\tilde{Q}_i} \leq \alpha$.

According to the Lebesgue differentiation theorem $f_{\tilde{Q}_i} \rightarrow f(x)$

for almost all $x \in F$ (whenever x is a Lebesgue point of f , because the family $\{\tilde{Q}_i\}$ is regular at x) and hence $f(x) \leq \alpha$ for almost all $x \in F$. $\square$

Corollary If f, α , and Ω are as above, then $|\Omega| \leq \alpha^{-1} \|f\|_1$.

Proof $|\Omega| = \sum_{k=1}^{\infty} |Q_k| \leq \sum_{k=1}^{\infty} \frac{1}{\alpha} \int_{Q_k} f \leq \alpha^{-1} \|f\|_1$. $\square$

Now we will apply the Calderón-Zygmund theorem to obtain an interesting (and important) estimate for the maximal function. ④

Recall that if $f \in L^1(\mathbb{R}^n)$, then

$$|\{Mf > t\}| \leq \frac{5^n}{t} \int_{\mathbb{R}^n} |f|,$$

but we also proved a better estimate

$$|\{Mf > t\}| \leq \frac{2 \cdot 5^n}{t} \int_{\{|f| > t/2\}} |f|.$$

Using the Calderón-Zygmund theorem we will prove

Proposition If $f \in L^1(\mathbb{R}^n)$, then

$$\frac{2^{-n} C(n)^{-1}}{t} \int_{\{|f| > C(n)t\}} |f| \leq |\{Mf > t\}| \leq \frac{2 \cdot 5^n}{t} \int_{\{|f| > \frac{t}{2}\}} |f|.$$

This result is quite surprising, because the lower bound for $|\{Mf > t\}|$ is very similar to the upper bound.

Proof Clearly, we only need to prove the first inequality, the left one.

Apply the Calderón-Zygmund decomposition ⑤
to $|f|$ with some $\alpha > 0$. The value of
 α will be fixed later.

If $\Omega = \bigcup_k Q_k$ is as in the Calderón-Zygmund
theorem, then

$$\alpha < \int_{Q_k} |f(x)| dx \stackrel{(*)}{\leq} 2^n \alpha \quad \text{for all } k.$$

Let l_k be the edge-length of Q_k .
Then for any $x \in Q_k$ we have

$$Q_k \subset B_x := B(x, \underbrace{\sqrt{n} l_k}_{\text{diam } Q_k}), \quad \text{so}$$

$$\alpha < \int_{Q_k} |f| \leq \frac{|B_x|}{|Q_k|} \int_{B_x} |f| \leq \omega_n n^{n/2} Mf(x), \quad x \in Q_k$$

Therefore,

$$|\{Mf > \alpha \omega_n^{-1} n^{-\frac{n}{2}}\}| \geq \sum_k |Q_k| \stackrel{(*)}{\geq}$$

$$\frac{2^{-n}}{\alpha} \sum_k \int_{Q_k} |f| = \frac{2^{-n}}{\alpha} \int_{\bigcup_k Q_k} |f| \geq \heartsuit$$

$|f| \leq \alpha$ a.e. in $\mathbb{R}^n \setminus \bigcup_k Q_k$, so up to a
set of measure zero

$\{ |f| > \alpha \} \subset \bigcup_k Q_k$ and hence

(6)

$$\heartsuit \geq \frac{2^{-n}}{\alpha} \int_{\{|f| > \alpha\}} |f|.$$

Taking $\alpha = t \omega_n n^{n/2} = c(n)t$
we obtain

$$|\{Mf > t\}| \geq \frac{2^{-n}}{c(n)t} \int_{\{|f| > c(n)t\}} |f|$$

The proof is complete.

□