Homework #2

Ex 17 We proved that a polynomial
$$P(z) = a_n z^n + \dots + a_1 z + a_0, \quad a_n \neq 0$$
defines a mapping
$$P : S^2 \longrightarrow S^2$$
of degree n. On the other hand P is homotopic to 0 which is a constant map
$$H(z,t) = t\,P(z).$$
Thus any polynomial P defines a map $P : S^2 \longrightarrow S^2$ of degree 0. Therefore
$$n = 0 \quad \text{for evey } n = 1, 2, 3, \dots$$
Whare is a mistake?

Ex 18 Prove that there is a map $f \in C^\infty(S^3, S^2 \times S^1)$ of degree 0 that is not homotopic to a constant map.

Hint: Use homotopy groups.

Ex 19 Let M^n be a compact connected oriented manifold without boundary. Prove that there is a mapping $f \in C^\infty(M^n, S^n)$ of degree 1.

Ex 20 Prove that every map $f \in C^\infty(S^n, \underbrace{S^1 \times \ldots \times S^1}_{n})$ has degree 0.

Ex 21 Using integration by parts prove that if $f \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} Jf(x)\,dx = 0$$

Ex 22 Prove that a vector field X is smooth if and only if it is smooth as a mapping between differentiable manifolds

$$X : M \to TM$$

Ex 23 Prove that the mappings

$$\tau \mapsto \tau^{-1} \quad \text{and} \quad (\sigma, \tau) \mapsto \sigma\tau$$

on a Lie group are smooth.

Ex 24 Prove that left-invariant vector fields are smooth.

Ex 25 Prove that $GL(n,\mathbb{R})$ is a Lie group with the Lie algebra isomorphic to $\mathfrak{g}(n,\mathbb{R})$.

$\boxed{Ex\ 26}$ Prove that the Heisenberg group $\mathbb{H}^1 = \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ with the group law

$$(z,t) * (z',t') = (z+z', t+t'+2\, \mathrm{im}\, z\overline{z'})$$

is a Lie group. Find a basis of left invariant vector fields. More precisely the dimension of Lie algebra is 3, so you need to find three left invariant vector fields (explicitly)

$$X_i = a_i(x,y,t)\frac{\partial}{\partial x} + b_i(x,y,t)\frac{\partial}{\partial y} + c_i(x,y,t)\frac{\partial}{\partial t}, \quad i=1,2,3,$$

that span the Lie algebra.

$\boxed{Ex\ 27}$ Provide an example of a non-complete vector field on $\mathbb{R}$.

$\boxed{Ex\ 28}$ Solve the exercise from p. 244

$\boxed{Ex\ 29}$ Find the third order expansion

$$\exp(tX)\exp(tY) = \exp\left(t(X+Y) + \frac{t^2}{2}[X,Y] + t^3(\,?\,) + O(t^4)\right)$$

and represent "?" in terms of commutators.

$\boxed{Ex\ 30}$ Solve the exercise from p. 278